

# Quasisymmetric functions from combinatorial Hopf monoids and Ehrhart Theory

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**Abstract.** We investigate quasisymmetric functions coming from combinatorial Hopf monoids. We show that these invariants arise naturally in Ehrhart theory, and that some of their specializations are Hilbert functions for relative simplicial complexes. This class of complexes, called forbidden composition complexes, also forms a Hopf monoid, thus demonstrating a link between Hopf algebras, Ehrhart theory, and commutative algebra. We also study various specializations of quasisymmetric functions.

**Resumé.** Nous étudions les fonctions quasisymétriques associées aux monoïdes de Hopf combinatoires. Nous démontrons que ces invariants sont des objets naturels à la théorie de Ehrhart. De plus, certains correspondent à des fonctions de Hilbert associées à des complexes simpliciaux relatifs. Cette classe de complexes, constitue un monode de Hopf, révélant ainsi un lien entre les algèbres de Hopf, la théorie de Ehrhart, et l'algèbre commutative. Nous étudions également diverses catégories de fonctions quasisymétriques.

**Keywords.** Chromatic Polynomials, Symmetric Functions, Combinatorial Species, Combinatorial Hopf Algebras, Ehrhart Theory, Hilbert functions

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## 1 Introduction

Chromatic polynomials of graphs, introduced by Birkhoff and Lewis (1946) are wonderful polynomials. Their properties can be understood as coming from three different theories:

1. Chromatic polynomials were shown by Beck and Zaslavsky (2006) to be Ehrhart functions for inside-out polytopes.
2. They arise as the Hilbert polynomial for the coloring ideal introduced by Steingrímsson (2001). Moreover, this ideal is the Stanley-Reisner module for a relative simplicial complex  $(\Gamma, \Delta)$ .
3. The chromatic polynomial is the image of a homomorphism from the incidence Hopf algebra of graphs, first studied in Schmitt (1994), to the polynomial algebra.

Similar results have been shown by Aguiar and Ardila (2010) for the Hopf algebra of generalized Permutohedra. Thus, we can give three distinct proofs of Stanley's Reciprocity Theorem of chromatic polynomials

Stanley (1973). Chromatic polynomials form a situation where ‘Ehrhart polynomial = Hilbert polynomial = polynomial coming from a Hopf algebra’. The idea of ‘Ehrhart = Hilbert’ has been studied before by Breuer and Dall (2010). We call such polynomials *triune*, because they can be studied from three different perspectives at one time.

The primary goal of this paper is to study *triune* quasisymmetric functions which are Ehrhart functions, specialize to Hilbert functions, and come from combinatorial Hopf algebras. The motivation is that such invariants have three different aspects, which give them a rich structure. Given any combinatorial Hopf monoid  $H$  with a Hopf submonoid  $K$ , there is a natural quasisymmetric function  $\Psi_K(\mathbf{h})$  associated to every element  $\mathbf{h} \in H$ . This invariant is a special case of the work of Aguiar et al. (2006). In our case, the invariant can be studied from the perspective of geometric combinatorics: there is a canonical relative simplicial complex  $(\Gamma_{K,\mathbf{h}}, \Delta_{\mathbf{h}})$  associated to  $\mathbf{h}$ , with a natural geometric realization in  $\mathbb{R}^I$ , such that  $\Psi_K(\mathbf{h})$  enumerates lattice points with positive coordinates inside of the complex. The resulting Ehrhart function is an Ehrhart quasisymmetric function as defined by Breuer and Klivans (2015). We show how principal specialization is a morphism of Hopf algebras to the ring of ‘Gaussian polynomial functions’, and that the corresponding Ehrhart ‘Gaussian polynomial’ is a Hilbert function of  $(\Gamma_{K,\mathbf{h}}, \Delta_{\mathbf{h}})$  with respect to a certain bigrading. Setting  $q = 1$  recovers known results.

The paper is organized as follows: we review definitions regarding the Coxeter complex of type  $A$ , and from Ehrhart theory. We discuss the relationship between Ehrhart theory and Hilbert functions for relative simplicial complexes  $(\Gamma, \Delta)$ , where  $\Delta$  is a subcomplex of the Coxeter complex, and define forbidden composition complexes. In Section 3, we review material on Hopf monoids, and define *triune* quasisymmetric functions, which are special cases of invariants defined by Aguiar et al. (2006). In Section 4, we show that forbidden composition complexes form the terminal Hopf monoid in the category of pairs of Hopf monoids, which implies that every *triune* quasisymmetric function is the Ehrhart quasisymmetric function for some canonical forbidden composition complex. Thus we have a link between geometric combinatorics and combinatorial Hopf algebras that was known only in special cases. In Section 5, we discuss various specializations of quasisymmetric functions from the Hopf algebra point of view. This is motivated by the lecture notes of Grinberg and Reiner (2015), which emphasize principal specialization at  $q = 1$ . This gives new combinatorial identities, including for Ehrhart polynomials. In the process, we discuss the notion of Gaussian polynomial function, which are linear combinations of polynomials in  $q$  with  $q$ -binomial coefficients.

## 2 Relative Composition complexes and Ehrhart Theory

The motivation for this work comes from the study of chromatic polynomials:

1. In Steingrímsson (2001), chromatic polynomials of graphs are shown to be Hilbert functions for coloring ideals, which is the Stanley-Reisner module for the relative coloring complex.
2. In Beck and Zaslavsky (2006), chromatic polynomials of graphs are shown to be Ehrhart polynomials of an inside-out polytope, which is the geometric realization of the relative coloring complex.

Thus, the Ehrhart polynomial of the inside-out polytope of a graph is the Hilbert polynomial of coloring ideal. We give a  $q$ -analogue of this result for arbitrary relative composition complexes.

A *set composition* is a sequence  $C_1, \dots, C_k$  of disjoint subsets of  $I$  such that  $\cup_{i=1}^k C_i = I$ . The *length* of the composition is  $\ell(C) = k$ . We denote set compositions with vertical bars, so  $12|3$  corresponds to the

set composition  $\{1, 2\}, \{3\}$ , and  $21|3 = 12|3$ . The sets  $C_i$  are *blocks*. Similarly, an integer composition  $\alpha$  is a sequence  $\alpha_1, \dots, \alpha_k$  of positive integers whose sum is  $n$ .

Given a set composition  $C$ , there is a natural flag of sets  $F(C) := S_1 \subset S_2 \subset \dots \subset S_{k-1} \subseteq S_k = I$ , where  $S_i = \cup_{j \leq i} C_j$ . Similarly, given such a flag  $F$ , there is a set composition  $C(F) := C_1, C_2, \dots, C_k$ , where  $C_i = S_i \setminus S_{i-1}$ . This is analogous to the classic situation for integer compositions, where there is a correspondence between integer compositions of length  $\Delta$  and subsets of  $[n]$  of size  $k - 1$ . We use both notations:  $S_i$  for the sets in the flag, and  $C_i$  for the blocks. The Coxeter complex of type  $A$  is the order complex on the boolean lattice  $2^I \setminus I$ . We let  $\Sigma_I$  denote the Coxeter complex of type  $A$  on the set  $I$ .

### 2.1 Ehrhart Quasisymmetric Function

Let  $x_1, \dots, x_i, \dots$  be a sequence of commuting indeterminates indexed by positive integers. A *quasisymmetric function* is a power series in  $x_1, \dots$ , whose terms have bounded degree, such that for any  $a_1, \dots, a_k$ , and  $i_1 < i_2 < \dots < i_k$  the coefficient of  $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$  is equal to the coefficient of  $x_1^{a_1} \dots x_k^{a_k}$ . A basis is given by the monomial quasisymmetric functions  $M_\alpha = \sum_{i_1 < \dots < i_k} \mathbf{x}^\alpha$  where  $\mathbf{x}^\alpha = x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$ .

Given a quasisymmetric function  $Q$ , and  $q \in \mathbb{K} \setminus \{0\}$ , and  $n \in \mathbb{N}$ , the *principal specialization*  $\mathbf{ps}(Q)(n)$  is given by  $\mathbf{ps}(Q)(n) = Q(1, q, q^2, \dots, q^{n-1}, 0, 0, 0, \dots)$ . For a fixed  $q$ , we view  $\mathbf{ps}(Q)$  as a function from  $\mathbb{N}$  to  $\mathbb{K}$ . When  $q = 1$ , we denote the specialization by  $\mathbf{ps}^1(Q)(n)$ . It is known that this is a polynomial function. The *stable principal specialization* is given by  $\mathbf{sps}(Q) = Q(1, q, q^2, \dots)$ . This gives a formal power series. However, it turns out that the coefficients  $Q(n)$  of the resulting power series is a quasi-polynomial in  $n$ .

Given a face  $F = \emptyset \subset S_1 \subset S_2 \subset \dots \subset S_m \subset I$  of  $\Sigma_I$ , there is a corresponding polyhedral cone in the positive orthant  $\mathbb{R}_{\geq 0}^I$ . The cone is given by the equations  $x_i < x_j$  whenever  $i \in S_k, j \notin S_k$  for some  $\Delta$ , and  $x_i = x_j$  whenever  $i \in S_k$  if and only if  $j \in S_k$ . For example, for the flag  $\{2, 4\} \subset \{1, 2, 4, 7\} \subset \{1, 2, 3, 4, 7, 9\}$ , we obtain the polyhedral cone given by  $x_2 = x_4 < x_1 = x_7 < x_3 = x_9 < x_5 = x_8$ . Thus, for any collection  $\mathcal{F}$  of faces of  $\Sigma_I$ , there is a collection  $C(\mathcal{F})$  of open polyhedral cones in  $\mathbb{R}_{\geq 0}^I$ . Given a lattice point  $\mathbf{a} \in \mathbb{R}_{> 0}^I$ , we let  $\mathbf{x}_\mathbf{a} = \prod_{i \in I} x_{a_i}$  be its monomial, where the coordinates of  $\mathbf{a}$  are encoded in the *indices*, not the exponents. The *Ehrhart quasisymmetric function* for  $C(\mathcal{F})$  is given by

$$E_{C(\mathcal{F})} = \sum_{\mathbf{a}} \mathbf{x}_\mathbf{a}$$

where the sum is over all lattice points which lie in some cone of  $C(\mathcal{F})$ . Since  $E_{C(\mathcal{F})} = \sum_{F \in \mathcal{F}} M_{\text{type}(C(F))}$ , this is a quasisymmetric function, first appearing in the work of Breuer (2015).

We mention specializations of  $E_{C(\mathcal{F})}$ , and their combinatorial interpretations. First,  $\mathbf{ps}^1(E_{C(\mathcal{F})})(n+1)$  is the number of lattice points in  $C(\mathcal{F}) \cap [0, n]^{|I|}$ . Also,  $[q^m] \mathbf{sps}(E_{C(\mathcal{F})})$  is the number of lattices points in  $C(\mathcal{F}) \cap \Gamma_m$ , where  $\Gamma_m$  is the simplex given by the equation  $\sum_{i \in I} a_i = m$ . Finally,  $[q^m] \mathbf{ps}(E_{C(\mathcal{F})})(n+1)$  is the number of lattice points in  $C(\mathcal{F}) \cap \Gamma_m \cap [0, n]^{|I|}$ .

### 2.2 Relative Composition complexes

We define Stanley-Reisner modules for relative simplicial complexes, and introduce relative composition complexes, which have a natural geometric realization as open polyhedral cones. We show that specializations of the Ehrhart quasisymmetric function for relative composition complexes give the Hilbert function of the Stanley-Reisner module.

A *relative simplicial complex* is a pair  $(\Gamma, \Delta)$  where  $\Gamma \subseteq \Delta$ , and  $\Delta$  is a simplicial complex. Given  $\Delta$  with vertices  $S$ , we let  $\mathbb{C}[S]$  be the polynomial ring with indeterminates  $s_1, \dots, s_k$ , the vertices of  $S$ . The *Stanley-Reisner ideal* for  $\Delta$  is generated by  $\langle \sigma \subseteq S : \sigma \notin \Delta \rangle$ , and the *Stanley-Reisner module* for  $(\Gamma, \Delta)$  is  $I_\Gamma/I_\Delta$ . The module is graded by total degree, and its Hilbert function  $H(\Gamma, \Delta)(n)$  is the number of monomials of degree  $n$  in the module. It is known that the Hilbert function is in fact a polynomial: details can be found in Stanley (1984).

We also extend the definition of double coning over a simplicial complex. Coning over a non-void complex  $\Delta$  consists of adding new vertex  $x$ , and adding new faces  $\sigma \cup \{x\}$  for all  $\sigma \in \Delta$ . If  $\Delta = \emptyset$ , we let  $\text{cone}(\Delta) = \emptyset$ . For  $(\Gamma, \Delta)$ ,  $\text{cone}(\Gamma, \Delta) = (\text{cone}(\Gamma), \text{cone}(\Delta))$ . Finally, the double cone is defined by  $\text{dcone}(\Gamma, \Delta) = \text{cone}(\text{cone}(\Gamma, \Delta))$ .

Now we define the simplicial complexes that are of interest to us. A *relative composition complex* is a relative complex  $(\Gamma, \Delta)$  where  $\Delta \subseteq \Sigma_I$ . Given a relative composition complex  $(\Gamma, \Delta)$ , a composition  $C$  of  $\Gamma$ , and a block  $B$  of  $C$ ,  $B$  is *forbidden* if every composition of  $\Delta$  that refines  $C$  does not contain  $B$  as block.  $(\Gamma, \Delta)$  is a *forbidden complex* if every composition of  $\Gamma$  either has a forbidden block, or is a facet of  $\Delta$ . While the definition seems unusual, we will see that forbidden composition complexes arise naturally in the study of Hopf monoids in species.

**Example 1** Here are some examples.

1. Let  $\Gamma_{k,I}$  consist of all set compositions that have at least one block of size  $\geq k$ . Then points in  $\Gamma_{k,I}$  consist of points in  $\mathbb{R}^I$  that have at least  $k$  equal coordinates. This arises in the study of the  $k$ -equal problem. We see that  $(\Gamma_{k,I}, \Sigma_I)$  is a forbidden composition complex.
2. Let  $I = \{a, b, c, d\}$ , and let  $\Delta$  be the complex with facets which correspond to the permutations  $abcd, abdc, adbc, adcb, dabc, dacb, dcab, dcba$ . Let  $\Gamma$  be the subcomplex with facets  $cd|b|a, cd|a|b, ab|c|d, ab|d|c, a|bc|d, a|cd|b, d|ab|c$ , and  $d|bc|a$ . Then  $(\Gamma, \Delta)$  is a forbidden composition complex. The Ehrhart quasisymmetric function is  $8M_{1111} + 4M_{112} + 2M_{121} + 2M_{211} + M_{22}$ .
3. Let  $\Gamma$  be the simplex corresponding to  $12|34|56$ . Then  $(\Gamma, \Sigma_{[6]})$  is a relative composition complex, but it is not a forbidden complex, because  $1|2|34|56, 12|3|4|56$  and  $|12|34|5|6$  are all faces of  $\Sigma_{[6]}$ . Thus,  $12|34|56$  has no forbidden blocks.

Forbidden composition complexes generalize coloring complexes. Given a graph  $g$ , let  $\Gamma_g$  denote the collection of set compositions  $C$  for which some block contains an edge of  $g$ . This is the coloring complex introduced by Steingrímsson (2001). We let  $(\Gamma_g, \Sigma_I)$  be the relative coloring complex. The Stanley-Reisner module for the double cone over  $(\Gamma_g, \Sigma_I)$  is the coloring ideal. Our relative coloring complex is thus an example of a forbidden composition complex.

In the case of relative composition complexes, the polynomial ring associated to  $\Delta$  has indeterminates given by all subsets  $S \subseteq I$ . We define the bidegree of  $S$  to be  $(|S|, 1)$ . In this case,  $H(\Gamma, \Delta)(m, n)$  is the function which counts the number of monomials of degree  $(m, n)$ .

**Theorem 2** Let  $(\Gamma, \Delta)$  be a relative composition complex. Then  $H(\text{dcone}(\Gamma, \Delta))(n) = \mathbf{ps}^1(E_{C_{\Delta \setminus \Gamma}})(n)$ . Similarly, letting  $H(\text{dcone}(\Gamma, \Delta))(q, n) = \sum_{m \geq 0} H(\text{dcone}(\Gamma, \Delta))(m, n)q^m$ , we have  $q^{n|I|}H(\text{dcone}(\Gamma, \Delta))(q^{-1}, n) = \mathbf{ps}(E_{C_{(\Delta \setminus \Gamma)}})(n)$ .

Our first result follows from work of Breuer and Klivans (2015). However, in their setting there is no natural Stanley-Reisner module. The second result is similar to work of Breuer and Dall (2010).

### 3 Hopf monoids and Characters

In this section, we discuss combinatorial Hopf monoids, their characters, and their quasisymmetric functions. Hopf monoids are a generalization of graphs, posets and matroids. The idea is that we have some notion of combinatorial structure, called a species, as introduced by Joyal (1981). Moreover, we have rules for combining and decomposing these structures in a coherent way. Hopf monoids in species were originally introduced in Aguiar and Mahajan (2010), although the variation we discuss here can be found in Aguiar and Mahajan (2013). Hopf monoids allow us to define a whole class of quasisymmetric functions, and prove identities relating quasisymmetric functions in the same class, such as the class of chromatic symmetric functions of graphs.

#### 3.1 Hopf monoids in species

**Definition 3** A species is an endofunctor  $F : \text{Set} \rightarrow \text{Set}$  on the category of finite sets with bijections. For each finite set  $I$ ,  $F_I$  is a finite set, and for every bijection  $\sigma : I \rightarrow J$  between finite sets, there is a bijection  $F_\sigma : F_I \rightarrow F_J$ , such that  $F_{\sigma \circ \tau} = F_\sigma \circ F_\tau$  for every pair  $\sigma : I \rightarrow J$ ,  $\tau : K \rightarrow I$ . It is connected if  $|F_\emptyset| = 1$ . All species in this paper are connected, and  $1_F$  denotes the only element of  $F_\emptyset$ .

The exponential generating function for  $F$  is  $F(x) = \sum_{n \in \mathbb{N}} |F_{[n]}| \frac{x^n}{n!}$ .

**Example 4** We list various examples of species.

1. The graph species  $G$ : the set  $G_I$  consists of all graphs with vertex set  $I$ . Given  $\sigma : I \rightarrow J$ , and  $g \in G_I$ ,  $G_\sigma(g) = h$  is the graph on vertex set  $J$  where  $i \sim j$  in  $h$  if and only if  $\sigma^{-1}(i) \sim \sigma^{-1}(j)$  in  $g$ . Then  $G(x) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!} = 1 + x + 2\frac{x^2}{2} + 8\frac{x^3}{6} + \dots$
2. The poset species  $P$ : the set  $P_I$  consists of all partial orders on  $I$ . Given  $\sigma : I \rightarrow J$ , and  $p \in P_I$ ,  $G_\sigma(p) = q$  is the partial order on  $J$  where  $i \leq_q j$  if and only if  $\sigma^{-1}(i) \leq_p \sigma^{-1}(j)$ .  $P(x) = 1 + x + 3\frac{x^2}{2} + 19\frac{x^3}{6} + \dots$
3. The matroid species, whose structures  $M_I$  consist of all matroids on  $I$ . Then  $M(x) = 1 + 2x + 5\frac{x^2}{2} + 16\frac{x^3}{6} + \dots$
4. The species  $R$ : the set  $R_I$  consists of all relative composition complexes  $(\Gamma, \Delta)$  where  $\Delta \subseteq \Sigma_I$ .

**Definition 5** A monoid is a species  $F$ , equipped with associative multiplication maps  $\mu_{S,T} : F_S \times F_T \rightarrow F_{S \sqcup T}$  for every pair  $S, T$  of finite sets, where  $S \sqcup T$  denotes disjoint union. We denote the product of  $f \in F_S$ ,  $g \in F_T$  by  $f \cdot g$ . Associativity means that  $(f \cdot g) \cdot h = f \cdot (g \cdot h)$  whenever the multiplication is defined. Moreover,  $1_F \cdot f = f = f \cdot 1_F$ .

**Example 6** We list various monoid operations.

1. The graph species  $G$  is a monoid. Given two graphs  $g$  and  $h$  with disjoint vertex sets,  $g \cdot h$  is their disjoint union: the graph with edges  $i \sim j$  if and only if  $i, j \in V(g)$  and  $i \sim j$  in  $g$ , or  $i, j \in V(h)$ , and  $i \sim j$  in  $h$ .
2. The poset species  $P$  is a monoid. The product is also given by disjoint union of partial orders.
3. The matroid species  $M$  is a monoid. The product is the direct sum.

4. The relative composition complex species  $\mathbf{R}$  is a monoid. Given  $(\Gamma, \Delta) \in \mathbf{R}_I$  and  $(\Gamma', \Delta') \in \mathbf{R}_J$ , we let  $\Delta \cdot \Delta'$  be the set of all quasi-shuffles of set compositions  $C, C'$ , where  $C \in \Delta$  and  $C' \in \Delta'$ . Here is an example: the quasi-shuffles of  $1|2$  and  $a|b$  are:  $1|2|a|b, 1|2a|b, 1|a|2|b, 1|a|2b, 1a|2|b, 1a|2b, a|1|2|b, a|1|2b, a|1b|2, a|1b|2, a|b|1|2, 1|a|b|2, 1a|b|2$ . The subcomplex  $\Gamma \cdot \Gamma'$  consists of all quasi-shuffles of all compositions  $C, C'$  where  $C \in \Gamma, C' \in \Delta'$  or  $C \in \Delta$  and  $C' \in \Gamma'$ . The product is then  $(\Gamma, \Delta) \cdot (\Gamma', \Delta') = (\Gamma \cdot \Gamma', \Delta \cdot \Delta')$ .

5. The species  $\Phi$  of forbidden composition complexes is a submonoid of  $\mathbf{R}$ .

**Definition 7** A combinatorial Hopf monoid (in species) is a monoid  $\mathbf{F}$  such that, for every  $S \subseteq I$ , there are partial functions restriction  $\mathbf{F}|_S : \mathbf{F}_I \rightarrow \mathbf{F}_S$  and contraction  $\mathbf{F}/S : \mathbf{F}_I \rightarrow \mathbf{F}_{I \setminus S}$ , subject to:

1. For any  $T \subseteq S \subseteq I$ ,  $(\mathbf{f}|_S)|_T = \mathbf{f}|_T$ .
2. For any  $T \subseteq S \subseteq I$ ,  $(\mathbf{f}/T)/S = \mathbf{f}/S$ .
3. For any  $T \subseteq S \subseteq I$ ,  $(\mathbf{f}|_S)/T = (\mathbf{f}/T)|_S$ .
4.  $(\mathbf{f} \cdot \mathbf{g})|_S = \mathbf{f}|_{S \cap A} \cdot \mathbf{g}|_{S \cap B}$ .
5.  $(\mathbf{f} \cdot \mathbf{g})/S = \mathbf{f}/(S \cap A) \cdot \mathbf{g}/(S \cap B)$ .

We are working with partial functions, so if one side of the equation is undefined, then so is the other side.

**Example 8** We list various examples of Hopf monoids.

1. The graph species  $\mathbf{G}$  is a combinatorial Hopf monoid. The restriction map  $\mathbf{g}|_S$  consists of the graph on  $S$  with edges  $i \sim j$  if and only if  $i \sim j$  in  $\mathbf{g}$ . In this case, we define  $\mathbf{g}/S = \mathbf{g}|_{I \setminus S}$ .
2. The poset species  $\mathbf{P}$  is a combinatorial Hopf monoid. Given  $\mathbf{p}$ , and  $S \subseteq I$ , we let  $\mathbf{p}[I]$  denote the induced subposet. Then  $\mathbf{p}|_S = \mathbf{p}[S]$  provided  $\mathbf{p}[S]$  is an order ideal of  $\mathbf{p}$ . Similarly, if  $\mathbf{p}[S]$  is a lower order ideal, we let  $\mathbf{p}/S = \mathbf{p}[I \setminus S]$ .
3. The matroid species  $\mathbf{M}$  is a combinatorial Hopf monoid, with the restriction and contraction.
4. The species  $\Phi$  of forbidden composition complexes is a Hopf monoid. We do not give the definition of the coproduct, as it is technical. It involves the notion of deconcatenation.

### 3.2 Characters and Inversion

Now we discuss characters of Hopf monoids.

**Definition 9** Given a Hopf monoid  $\mathbf{H}$ , and a field  $\mathbb{K}$ , a character is a multiplicative function  $\varphi : \mathbf{H} \rightarrow \mathbb{K}$ . For every finite set  $I$ , there is a map  $\varphi_I : \mathbf{H}_I \rightarrow \mathbb{K}$ , natural in  $I$ , such that, for all  $I = S \sqcup T$ ,  $\mathbf{h}_S \in \mathbf{H}_S, \mathbf{h}_T \in \mathbf{H}_T$ , we have  $\varphi_S(\mathbf{h}_S)\varphi_T(\mathbf{h}_T) = \varphi_I(\mathbf{h}_S \cdot \mathbf{h}_T)$ . The character is connected if  $\varphi_\emptyset(\mathbf{h}_\emptyset) = 1$ .

**Example 10** One example is the character given by  $\varphi_I(\mathbf{h}) = 1$  for all  $I, \mathbf{h} \in \mathbf{H}_I$ . This is the zeta character.

Let  $\mathbf{G}$  be the Hopf monoid of graphs. Given a graph  $\mathbf{g}$ , let

$$\varphi(\mathbf{g}) = \begin{cases} 1 & \mathbf{g} \text{ has no edges} \\ 0 & \text{otherwise} \end{cases}$$

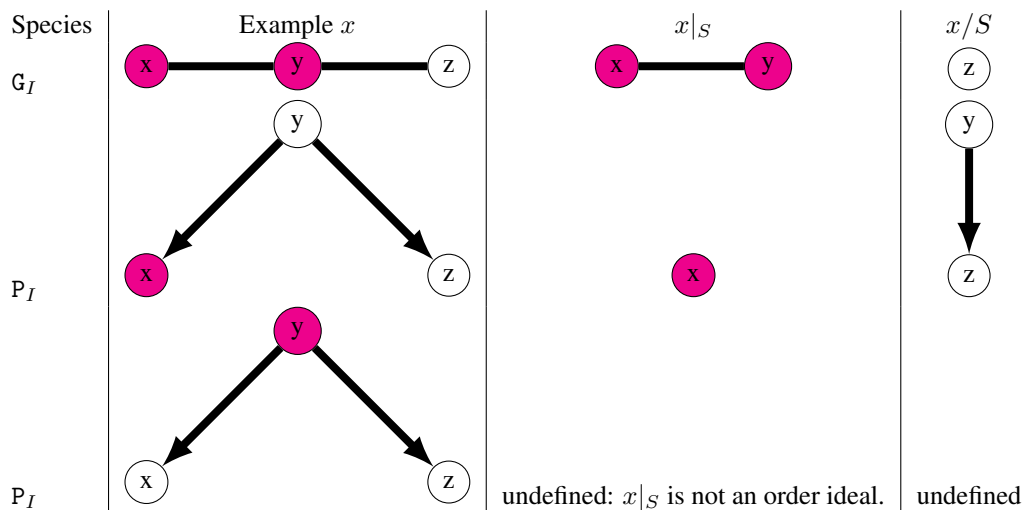


Fig. 1: Examples of restriction and quotient. Shaded vertices are elements of  $S$ .

The set  $\chi(H)$  of connected characters on  $H$  is a group, with multiplication given by:

$$(\varphi * \psi)_I(\mathbf{h}) = \sum_{S \subseteq I} \varphi_S(\mathbf{h}|_S) \psi_{I \setminus S}(\mathbf{h}/S)$$

for  $\varphi, \psi \in \chi(H)$ , where the right hand side is 0 for any  $S$  where  $\mathbf{h}|_S$  or  $\mathbf{h}/S$  is undefined.

The inverse of a character  $\varphi$  is defined recursively:

1.  $\varphi_\emptyset^{-1} = \varphi_\emptyset$
2. For  $\mathbf{h} \in H_I$ ,  $\varphi^{-1}(\mathbf{h}) = - \sum_{S \subset I} \varphi_S^{-1}(\mathbf{h}|_S) \varphi_{I \setminus S}(\mathbf{h}/S)$

We discuss characters coming from Hopf submonoids  $K \subseteq H$ . A *Hopf submonoid*  $K$  is a subspecies, meaning that  $K_I \subseteq H_I$  for all  $I$ . Moreover, the product, restriction, and contraction of elements of  $K$  remain in  $K$ . Given a submonoid  $K \subset H$ , there is a character  $\varphi_K : H \rightarrow \mathbb{K}$  given by:

$$\varphi_K(\mathbf{h}) = \begin{cases} 1 & \mathbf{h} \in K_I \\ 0 & \text{otherwise} \end{cases}$$

In the case of graphs, one Hopf submonoid is the species of edgeless graphs. In the case of posets, there is the Hopf submonoid of antichains. There is a Hopf monoid of generalized Permutohedra, and the character Aguiar and Ardila (2010) study also comes from a Hopf submonoid. Finally, the Hopf monoid of composition complexes is a Hopf submonoid of  $\mathfrak{F}$ . In each of these cases, we obtain a character.

### 3.3 The quasisymmetric function associated to a character

We recall the quasisymmetric function associated to a character on a Hopf monoid  $H$ . Given a set composition  $C$  of  $I$ , and  $\mathbf{h} \in H_I$ , define  $\varphi_C(h_1, \dots, h_k) = \prod_{i=1}^k \varphi(h_i)$ ,  $h_i = (h|_{S_i})/S_{i-1}$ , where  $S_i \in F(C)$ .

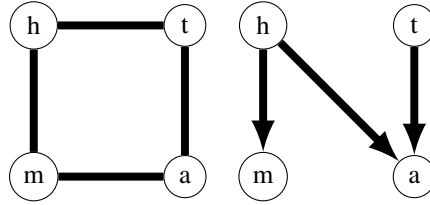


Fig. 2: Example graph and poset

**Definition 11** Given a combinatorial Hopf monoid  $\mathbb{H}$ , a character  $\varphi$ , a finite set  $I$ , and  $\mathbf{h} \in \mathbb{H}_I$ , define

$$\Psi_\varphi(\mathbf{h}) = \sum_{C \models I} \left( \prod_{i=1}^{\ell(C)} \varphi(\mathbf{h}|_i) \right) M_{\text{type}(C)}.$$

Given a combinatorial Hopf monoid, the vector space generated by the equivalence classes of  $\mathbb{H}$ -structures forms a combinatorial Hopf algebra, which appears in Aguiar and Mahajan (2010). Moreover, by work of Aguiar et al. (2006), there is a unique morphism from this Hopf algebra to  $QSym$ . Our definition  $\Psi_\varphi(\mathbb{H})$  is the resulting map.

There is a description for  $\Psi$  in terms of colorings. Given a coloring  $f : I \rightarrow \mathbb{N}$ , and  $i \in \mathbb{N}$ , we let  $\mathbf{h}|_i = \mathbf{h}|_{f^{-1}(\{i\})} / f^{-1}(\{i-1\})$  be the  $i$ th minor of  $\mathbf{h}$  under  $f$ .

**Theorem 12** Let  $\mathbb{H}$  be a combinatorial Hopf monoid, with a character  $\varphi : \mathbb{H} \rightarrow \mathbb{E}$ . Fix a finite set  $I$ , and  $\mathbf{h} \in \mathbb{H}_I$ . Then

$$\Psi_\varphi(\mathbf{h}) = \sum_{f: I \rightarrow \mathbb{N}} \varphi_f(\mathbf{h}) \mathbf{x}_f$$

where  $\varphi_f(\mathbf{h}) = \prod_{i \in \mathbb{N}} \varphi(\mathbf{h}|_i)$ , which is well-defined.

For a coloring of a graph  $\mathbf{g}$ , the  $i$ th minor is the induced subgraph on the  $i$ th color class, so  $\varphi(\mathbf{g}|_i) = 1$  if and only if the  $i$ th color class is an independent set. Thus our quasisymmetric function enumerates proper colorings, giving the chromatic symmetric function introduced by Stanley (1995). For example, for the graph in figure 3.2 the resulting chromatic symmetric function is  $24M_{1111} + 4M_{211} + 4M_{121} + 4M_{112} + 2M_{22}$ . For posets,  $\varphi_f(\mathbf{p}) = 1$  if and only if  $f : I \rightarrow \mathbb{N}$  is a strictly order preserving map, which is the quasisymmetric function for strict  $P$ -partitions considered by Stanley (1972). For example, for the poset  $p$  in figure 3.2, the quasisymmetric function is given by  $5M_{1111} + 2M_{211} + M_{121} + 2M_{112} + M_{22}$ .

**Theorem 13** Let  $\mathfrak{K}$  be the Hopf monoid of forbidden composition complexes. Then for all  $(\Gamma, \Delta) \in \mathfrak{K}_I$ , we have  $\Psi_\varphi(\Gamma, \Delta) = E_{C(\Delta \setminus \Gamma)}$ .

## 4 Forbidden composition complexes

We show that, for any pair  $\mathbb{K} \subseteq \mathbb{H}$  of Hopf monoids, and any element  $\mathbf{h} \in \mathbb{H}_I$ , there is a forbidden composition complex  $(\Gamma_{\mathbb{K}, \mathbf{h}}, \Delta_{\mathbf{h}})$  whose Ehrhart quasisymmetric function is  $\Psi_{\varphi_{\mathbb{K}}}(\mathbf{h})$ . Let  $\mathbb{H}$  be a combinatorial Hopf monoid,  $\mathbf{h} \in \mathbb{H}_I$ , and define  $\Delta_{\mathbf{h}} \subseteq \Sigma$  to be those faces  $F$  such that  $h_i$  is defined for all  $S_i \in F$ . This defines a morphism of Hopf monoids  $\Delta : \mathbb{H} \rightarrow \mathbb{C}$ , the species of composition complexes.



Let  $K \subseteq H$  be a Hopf submonoid, and define  $\Delta_{K,h}$  to consist faces  $F \in \Delta_h$  such that some minor  $h|_i \notin K_{S_i - S_{i-1}}$ . Since  $K$  is a Hopf submonoid,  $\Gamma_{K,h}$  is a simplicial complex. Moreover,  $(\Gamma_{K,h}, \Delta_h)$  is a forbidden composition complex, and the map  $\Gamma_K : H \rightarrow \mathfrak{F}$  defined by  $\Gamma(h) = (\Gamma_{K,h}, \Delta_h)$  is a morphism of Hopf monoids. However, even more is true: the quasisymmetric function  $\Psi_{\varphi_K}(h)$  is the Ehrhart quasisymmetric function for  $\Gamma(h)$ :

**Theorem 14** *Given a set  $I$ , let  $\mathfrak{F}_I$  denote the set of all forbidden composition complexes on  $I$ , and  $C_I$  denote the set of all composition complexes on  $I$ .*

1. *Given any combinatorial Hopf monoid  $H$ , with  $K \subseteq H$ , there exists unique morphisms of combinatorial Hopf monoids  $\Delta : K \rightarrow C$ ,  $\Gamma_K : H \rightarrow \mathfrak{F}$  such that  $\iota\Delta = \Gamma_{K,\iota}$ , where  $\iota$  denotes inclusion maps  $K \subseteq H$ , and  $C \subseteq \mathfrak{F}$ .*
2. *Under this map,  $\Psi_{\varphi_K}(h) = E(C(\Gamma(h)))$ .*

For a graph  $g$ ,  $\Delta_g$  is the relative coloring complex. Given a poset  $p$ , let  $C(p)$  be the polyhedral cone in  $\mathbb{R}^I$  bounded by equations  $x_i \leq x_j$ , for all  $i \leq j$  in  $p$ . Then  $\Delta_p$  consists of all cones in the Coxeter arrangement which lie in  $C(p)$ . Similarly,  $\Gamma_{K,p}$  consists of the cones which lie on the boundary of  $C(p)$ .

## 5 Specializations

We discuss specializations of quasisymmetric functions, and interpretations of  $\Psi$  under specialization. We show combinatorial identities relating quasisymmetric functions for various elements of the same combinatorial Hopf monoid. It is known that  $\mathbf{ps}^1$  is a Hopf algebra homomorphism from  $QSym$  to  $\mathbb{K}[x]$ . We show that  $\mathbf{ps}$  is a morphism of Hopf algebras in general. The image of  $\mathbf{ps}$  is the ring of Gaussian polynomial functions, which are  $q$ -analogues of polynomials. We also study the stable principal specialization  $\mathbf{psp}$ . While this section primarily emphasizes the Hopf algebra perspective, many of the results are of combinatorial interest.

### 5.1 Gaussian polynomials and principal specialization

Clearly,  $\mathbf{ps}(Q) : \mathbb{N} \rightarrow \mathbb{C}$  is a polynomial function when  $q = 1$ . This leads to the question of what type of function we get for general  $q$ . For now, assume that we are working over  $\mathbb{C}(q)$ .

For any integer  $m$ , define  $D_m(f) : \mathbb{N} \rightarrow \mathbb{C}(q)$  by  $D_m(f)(n) = f(n+1) - q^m f(n)$ , and  $D^m(f) = D_m \circ D^{m-1}(f)$ . A function  $f$  is a *Gaussian polynomial function* of degree at most  $d$  if  $D^{d+1}(f) = 0$ . We recover the classical definitions when  $q = 1$ . The terminology comes from the fact that  $q$ -binomial coefficients are sometimes called Gaussian polynomials, and all Gaussian polynomial functions can be expressed as linear combinations of  $q$ -binomial coefficients. Consider a Gaussian polynomial function of degree  $m$ . Then we can define  $f(-n) = q^{-m}(f(-n+1) - D_m(f)(-n))$ , for  $n > 0$ . Thus Gaussian polynomials are functions from  $\mathbb{Z} \rightarrow \mathbb{C}$ .

**Theorem 15** *The algebra of Gaussian polynomials,  $G$ , is a Hopf algebra, with basis given by  $[x]^n$ ,  $n \in \mathbb{N}$ . The unit is 1, and multiplication is given by  $[x]^k \cdot [x]^m = [x]^{k+m}$ . The comultiplication sends  $[x]$  to  $[x] \otimes 1 + q^x \otimes [x]$ , and the antipode is generated by  $S([x]) = [-x]$ .*

*Moreover,  $\mathbf{ps} : QSym \rightarrow G$  is a morphism of Hopf algebras, and  $G$  is graded as an algebra, but not as a coalgebra.*

Let us consider some examples. For the graph  $g$  in figure 3.2, the resulting Gaussian chromatic polynomial is  $14q^6 \begin{bmatrix} n \\ 4 \end{bmatrix}_q + (2q^5 + 4q^4 + 2q^3) \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + 2q^2 \begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$ . For the poset in figure 3.2, the resulting Gaussian polynomial is  $q^6 \begin{bmatrix} n \\ 4 \end{bmatrix}_q + (q^5 + q^4 + q^3) \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + q^2 \begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$ . Finally, for the forbidden composition complex in Example 1, part 2, the resulting Gaussian polynomial is  $2q^6 \begin{bmatrix} n \\ 4 \end{bmatrix}_q + (3q^5 + q^4 + q^3) \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + q^2 \begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$ .

**Definition 16** Given a combinatorial Hopf monoid  $H$  with a character  $\varphi$ , and  $h \in H_I$ , define  $P_\varphi(h, q, n) = ps(\Psi_\varphi(h))$ . This is the polynomial of  $h$  associated to  $\varphi$ . Alternatively,  $P_\varphi(h, q, n) = \sum_{f:I \rightarrow [n]} \varphi_f(h)q^{w(f)}$ , where  $w(f) = \sum_{i \in I} (f(i) - 1)$ .

For a poset  $p$ ,  $[q^n]P_\varphi(p, q, m)$  is the number of  $p^*$ -partitions of  $n$  with part size at most  $m$ .

**Proposition 17** Let  $H$  be a combinatorial Hopf monoid with character  $\varphi$ , and let  $h \in H_I, k \in H_J$ , where  $I$  and  $J$  are disjoint sets. Then the following identities hold:

1.  $P_\varphi(h \cdot k, q, n) = P_\varphi(h, q, n) \cdot P_\varphi(k, q, n)$ .
2. for any  $n, m \in \mathbb{N}$ ,  $P_\varphi(h, q, m+n) = q^{m|I|} \sum_{S \subseteq I} q^{-m|S|} P_\varphi(h|_S, q, m) \cdot P_\varphi(h/S, q, n)$ .
3.  $P_\varphi(h, q, -n) = q^{|I|} P_{\varphi^{-1}}(h, q^{-1}, n)$ .

The last identity is a reciprocity result. On the left-hand side, we are counting negative colors, so we expect to have negative powers of  $q$ . When  $q = 1$ , these identities are already known for graphs and posets. Also, combinatorial reciprocity for  $P_\varphi(p, q, n)$  is also due to Stanley (1972).

When  $\varphi = \varphi_K$  for some Hopf submonoid  $K \subseteq H$ , then the fact that  $P_{\varphi_K}(h, 1, n)$  is an Ehrhart function, and a Hilbert function, allows us to conclude new results regarding  $\varphi^{-1}$ .

**Corollary 18** Let  $K \subseteq H$  be an inclusion of Hopf monoids, and let  $h \in H_I$ . Then  $(-1)^{|I|} P_{\varphi_K^{-1}}(h, 1, n) = \sum_{\mathbf{a} \in \mathbb{Z}^I \cap (0, n]^I} w(\mathbf{a})$ , where  $w(\mathbf{a})$  is the number of cones  $C$  of  $(\Gamma_{K,h}, \Delta_h)$  such that  $\mathbf{a} \in \bar{C}$ .

Moreover,  $\varphi^{-1}(h) = \chi(\Gamma_{K,h}, \Delta_h) = \sum_{\sigma \in \Delta_h \setminus \Gamma_{K,h}} (-1)^{|\sigma|}$ , the Euler characteristic.

### 5.2 The stable principal specialization

We define  $Q_\varphi(h, q) = sps(\Psi_\varphi(h))$ . In particular,  $Q_\varphi(h, q) = \sum_{f:I \rightarrow \mathbb{N}} \varphi_f(h)q^{w(f)}$ . For posets,  $Q_\varphi(p, q)$  is the generating function for strict  $p^*$ -partitions.

**Theorem 19** Let  $H$  be a combinatorial Hopf monoid with character  $\varphi$ , and let  $h \in H_I, k \in H_J$ , where  $I$  and  $J$  are disjoint sets. Then the following identities hold:

1.  $Q_\varphi(h \cdot k, q) = Q_\varphi(h, q) \cdot Q_\varphi(k, q)$ .
2.  $Q_\varphi(h, q^{-1}) = (-q)^{|I|} Q_{\varphi^{-1}}(h, q)$ .

The last identity is a reciprocity result, and is due to Stanley (1972) in the case of posets.

## 6 Conclusion

We conclude with questions:

1. Which properties of complexes are stable under the Hopf monoid operations in  $\Phi$ ? Do shellable complexes form a Hopf submonoid? What about Cohen-Macaulay complexes, or partitionable complexes?
2. What properties of a forbidden composition complex allow us to conclude that the triune quasisymmetric function is positive in the basis of fundamental quasisymmetric functions? This question is interesting: The complex  $(\Gamma, \Delta)$  in Example 1 part 2 has the feature that the Ehrhart quasisymmetric functions of  $\Gamma$  and  $\Delta$  are not  $F$ -positive, but the triune quasisymmetric function for  $(\Gamma, \Delta)$  is  $F$ -positive.
3. If we linearize  $\Phi$ , what other natural bases does it possess?

Forbidden composition complexes, and triune quasisymmetric functions merit further study, as these geometric objects and their symmetric function invariants can be approached from three distinct perspectives.

## References

- M. Aguiar and F. Ardila. The hopf monoid of generalized permutohedra. in preparations, slides available, 2010.
- M. Aguiar and S. Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. ISBN 978-0-8218-4776-3. With forewords by Kenneth Brown and Stephen Chase and André Joyal.
- M. Aguiar and S. Mahajan. Hopf monoids in the category of species. *Contemporary Mathematics*, 585: 17–124, 2013. doi: <http://dx.doi.org/10.1090/conm/585>.
- M. Aguiar, N. Bergeron, and F. Sottile. Combinatorial Hopf algebras and generalized Dehn-Sommerville relations. *Compos. Math.*, 142(1):1–30, 2006. ISSN 0010-437X. doi: 10.1112/S0010437X0500165X. URL <http://dx.doi.org/10.1112/S0010437X0500165X>.
- M. Beck and T. Zaslavsky. Inside-out polytopes. *Adv. Math.*, 205(1):134–162, 2006. ISSN 0001-8708. doi: 10.1016/j.aim.2005.07.006. URL <http://dx.doi.org/10.1016/j.aim.2005.07.006>.
- G. D. Birkhoff and D. C. Lewis. Chromatic polynomials. *Trans. Amer. Math. Soc.*, 60:355–451, 1946. ISSN 0002-9947.
- F. Breuer. An invitation to Ehrhart theory: polyhedral geometry and its applications in enumerative combinatorics. In *Computer algebra and polynomials*, volume 8942 of *Lecture Notes in Comput. Sci.*, pages 1–29. Springer, Cham, 2015. doi: 10.1007/978-3-319-15081-9\_1. URL [http://dx.doi.org/10.1007/978-3-319-15081-9\\_1](http://dx.doi.org/10.1007/978-3-319-15081-9_1).

- F. Breuer and A. Dall. Viewing counting polynomials as Hilbert functions via Ehrhart theory. In *22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*, Discrete Math. Theor. Comput. Sci. Proc., AN, pages 545–556. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010.
- F. Breuer and C. Klivans. Scheduling problems. arXiv:1401.2978, 2015.
- D. Grinberg and V. Reiner. Hopf algebras in combinatorics. arXiv:1409.8356, 2015.
- A. Joyal. Une théorie combinatoire des séries formelles. *Adv. in Math.*, 42(1):1–82, 1981. ISSN 0001-8708. doi: 10.1016/0001-8708(81)90052-9. URL [http://dx.doi.org/10.1016/0001-8708\(81\)90052-9](http://dx.doi.org/10.1016/0001-8708(81)90052-9).
- W. R. Schmitt. Incidence Hopf algebras. *J. Pure Appl. Algebra*, 96(3):299–330, 1994. ISSN 0022-4049. doi: 10.1016/0022-4049(94)90105-8. URL [http://dx.doi.org/10.1016/0022-4049\(94\)90105-8](http://dx.doi.org/10.1016/0022-4049(94)90105-8).
- R. P. Stanley. *Ordered structures and partitions*. American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 119.
- R. P. Stanley. Acyclic orientations of graphs. *Discrete Math.*, 5:171–178, 1973. ISSN 0012-365X.
- R. P. Stanley. An introduction to combinatorial commutative algebra. In *Enumeration and design (Waterloo, Ont., 1982)*, pages 3–18. Academic Press, Toronto, ON, 1984.
- R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995. ISSN 0001-8708. doi: 10.1006/aima.1995.1020. URL <http://dx.doi.org/10.1006/aima.1995.1020>.
- E. Steingrímsson. The coloring ideal and coloring complex of a graph. *J. Algebraic Combin.*, 14(1):73–84, 2001. ISSN 0925-9899. doi: 10.1023/A:1011222121664. URL <http://dx.doi.org/10.1023/A:1011222121664>.