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Order Filter Model for Minuscule Plücker Relations[†]

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Abstract. The Plücker relations which define the Grassmann manifolds as projective varieties are well known. Grassmann manifolds are examples of minuscule flag manifolds. We study the generalized Plücker relations for minuscule flag manifolds independent of Lie type. To do this we combinatorially model the Plücker coordinates based on Wildberger's construction of minuscule Lie algebra representations; it uses the colored partially ordered sets known as minuscule posets. We obtain, uniformly across Lie type, descriptions of the Plücker relations of "extreme weight". We show that these are "supported" by "double-tailed diamond" sublattices of minuscule lattices. From this, we obtain a complete set of Plücker relations for the exceptional minuscule flag manifolds. These Plücker relations are straightening laws for their coordinate rings.

Résumé. Les relations de Plücker qui définissent les variétés de Grassmann comme variétés projectifs sont bien connus. Les variétés de Grassmann sont des exemples de variétés de drapeaux minuscules. Nous étudions les relations de Plücker généralisées pour variétés de drapeaux minuscules indépendantes de type. Pour ceci, nous créons une modélisation des coordonnées de Plücker basée sur la construction de Wildberger des représentations minuscules d'algèbres de Lie; elle utilise les ensembles ordonnés et colorés qu'on appelle "posets minuscules". À travers ce modèl, nous obtenons une description uniforme des relations de Plücker de "poids extrêmes". Nous montrons que ces relations sont supportés par un sous-réseau des réseaux minuscules . De cette façon, nous obtenons un ensemble complet de relations de Plücker pour les cas de types exceptionnels. Ces relations de Plücker sont des "lois de redressage" pour leurs anneaux coordonnées.

Keywords. Plücker relations, minuscule flag manifolds, minuscule posets, minuscule representations

1 Introduction

The Grassmann manifold $Gr(d, n)$ is the complex manifold which consists of the d-dimensional complex subspaces of the vector space $\mathbb{C}^n$. It has a standard projective embedding; the homogeneous coordinates for this embedding are usually indexed by the naturally ordered collection of d-element subsets of $\{1, 2, \ldots, n\}$. The quadratic relations among these coordinates have a nice combinatorial formulation using this index set. More generally, each of the "minuscule" flag manifolds also has a standard projective embedding for which the homogeneous coordinates are indexed by a natural partially ordered set.

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Such ordered coordinates are called Plücker coordinates; the minuscule Plücker relations are the quadratic relations among them. The minuscule flag manifolds are the Grassmann manifolds (Lie type A), the maximal orthogonal Grassmannians (types B and D), the even quadrics (also type D), and two "exceptional" manifolds: the complex Cayley plane (type E_6) and the Freudenthal variety (type E_7). We seek a uniform combinatorial description of the quadratic Plücker relations for the minuscule flag manifolds that is independent of Lie type. Our main results can be summarized as follows:

Theorem 1 *The "extreme weight" minuscule Plücker relations are "standard straightening laws" on "double-tailed diamond sublattices" of the Plücker coordinates. These are all of the Plücker relations for the complex Cayley plane (type E_6). For the Freudenthal variety (type E_7) we obtain a complete set of Plücker relations by supplementing the extreme weight relations with seven "zero weight" relations.*

This theorem describes a certain kind of minuscule Plücker relation. This is apparently the first time that all of the Plücker relations have been produced for the two exceptional minuscule flag manifolds. Our approach uses minuscule posets and lattices. These objects were introduced in Proctor (1984); they encode the weights of related minuscule Lie algebra representations. The simplest nontrivial minuscule lattices correspond to the natural representations of the even orthogonal algebras $\mathfrak{o}(2n)$. In these cases the Hasse diagram of the minuscule lattice is a "double-tailed diamond." We show that the structure of the relations found in these model cases is also possessed by the extreme Plücker relations for the other minuscule cases.

The known properties of the coordinate ring of the standard embedding for the Grassmann manifold inspired the following definition by Eisenbud in 1977:

Definition 2 *Suppose R is a ring, A an R-algebra, and H a finite poset contained in A which generates A as an R-algebra. A* standard monomial *of A is the product of a chain in H, i.e. an element of the form $a_1a_2\dots a_k$ with $a_i \in H$ for each $1 \le i \le k$ and $a_1 \preceq a_2 \preceq \cdots \preceq a_k$.*

Suppose $a, b \in H$ are an incomparable pair and suppose $ab = \sum_{i=1}^{s} r_i h_{i1} h_{i2} \dots h_{ik_i}$ is an expression for ab as a linear combination of standard monomials. Such a relation is called a straightening law *if $h_{i1} \preceq a$ and $h_{i1} \preceq b$ for every $1 \le i \le s$.*

A result of Seshadri implied that the Plücker relations for all minuscule flag manifolds are determined by straightening laws, but its proof did not explicitly produce the straightening laws:

Theorem 3 (Seshadri (1978)) *In the coordinate ring for a minuscule flag manifold under its standard embedding, the standard monomials on the Plücker coordinates form a basis. The straightening laws on these Plücker coordinates form a basis for its quadratic minuscule Plücker relations.*

The poset of standard minuscule Plücker coordinates is actually a lattice, meaning it has a meet ($\wedge$) and a join ($\vee$) operation. Lakshmibai and Gonciulea used the lattice structure to give relations for a certain flat degeneration of the manifold. Let H be the set of Plücker coordinates for a minuscule flag manifold X under its standard embedding.

Theorem 4 (Gonciulea and Lakshmibai (1996)) *The minuscule flag variety X degenerates flatly to the toric variety defined by the relations $\{ab = (a \wedge b)(a \vee b)\}_{a,b \in H}$.*

Chirivì and Maffei (2013) studied the Plücker relations for the maximal orthogonal Grassmannians by using Pfaffians to model the Plücker coordinates. They gave a "straightening algorithm" for the coordinate ring using relations among these Pfaffians. This parallels a common straightening algorithm for the Grassmann manifolds. Since our type-independent results only give "extreme weight" Plücker relations, in general those straightening algorithms produce many more relations for their respective manifolds. Chirivì, Littelmann, and Maffei (2009) briefly considered the Plücker relations of the Freudenthal variety. There they gave a single relation which can in principle be used to generate all of the Plücker relations with a Lie algebra action. Our results for this variety go much further.

The coordinate ring of a minuscule flag manifold under its standard embedding is known to be defined by its quadratic Plücker relations, which are determined by straightening laws on its Plücker coordinates. We describe the most accessible of these straightening laws, which are of the same form as the single straightening law for a model $\mathfrak{o}(2n)$ example. This straightening law is "supported" by a double-tailed diamond lattice. In particular, the first standard monomial in our straightening law is the product of the meet and join of the incomparable pair that appeared in Gonciulea and Lakshmibai (1996). In contrast to that paper, we obtain all of the terms in the straightening law for the extreme weight Plücker relations.

2 Formulation of the Plücker problem

Fix a simple complex Lie algebra $\mathfrak{g}$ of rank n with Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ denote the set of roots. Let S denote the nodes of the Dynkin diagram as indexed in Humphreys (1972). Let $\{h_\alpha\}_{\alpha\in\Phi}$ denote the coroots in $\mathfrak{h}$, and let $\langle \lambda, \alpha\rangle := \lambda(h_\alpha)$ be the application of a weight λ to the coroot for α. Let $\{\alpha_i\}_{i\in S}$ in Φ and $\{\omega_i\}_{i\in S}$ in $\mathfrak{h}^*$ denote the simple roots and the fundamental weights. Let W be the Weyl group of Φ, which acts on $\mathfrak{h}^*_{\mathbb{R}}$ by reflection over the root hyperplanes. It is generated by simple reflections $\{s_i\}_{i\in S}$ There is a standard partial order on the set of weights: For weights $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \preceq \lambda$ if $\lambda - \mu$ is a nonnegative integral sum of the simple roots. A weight λ is said to be dominant integral if $\langle \lambda, \alpha\rangle$ is a nonnegative integer for every positive root α. Given a dominant integral weight λ, let V_λ denote an irreducible $\mathfrak{g}$-module of highest weight λ. Every finite dimensional irreducible $\mathfrak{g}$-module is isomorphic to V_λ for some dominant integral weight λ.

Definition 5 *A dominant integral weight $\lambda \in \mathfrak{h}^*$ is* minuscule *if every weight of the irreducible $\mathfrak{g}$-module V_λ lies in the Weyl group orbit of λ. The following is the complete list of minuscule weights by Lie type:*

$$A_n\text{: } \omega_1, \ldots, \omega_r; \quad B_n\text{: } \omega_n; \quad C_n\text{: } \omega_1; \quad D_n\text{: } \omega_1, \omega_{n-1}, \omega_n; \quad E_6\text{: } \omega_1, \omega_6; \quad E_7\text{: } \omega_7.$$

Given a minuscule weight λ, a module isomorphic to V_λ is called a minuscule representation of $\mathfrak{g}$. It is small in the following sense: Every finite dimensional highest weight module has a non-empty weight space for each weight in the Weyl group orbit of its highest weight. In a minuscule representation, there are no other weights. The weights of any finite-dimensional $\mathfrak{g}$-module V_λ form a finite distributive lattice which we denote L_λ. Any weight basis of a minuscule $\mathfrak{g}$-module is in bijection with the lattice L_λ of their weights. Hence, we focus our study of minuscule representations on the lattice of weights L_λ.

For each minuscule weight there is a standard projective embedding of an associated minuscule flag manifold. The Plücker coordinates are given by a weight basis of the minuscule $\mathfrak{g}$-module V_λ, so they are ordered by the lattice L_λ. By a theorem of Kostant, the Plücker relation problem for the resulting coordinate ring can be stated entirely in the language of representation theory: Let v be a highest weight vector of V_λ. The $\mathfrak{g}$-module $Sym^2(V_\lambda)$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{g}).(v)^2 \oplus I$ of $\mathfrak{g}$-modules for a unique submodule I.

Problem 6 *Each Plücker relation for the coordinate ring of the flag manifold embedded by λ is given by the vanishing of a nonzero vector of I. Find a spanning set (or basis) for I.*

3 Minuscule posets

For a simply laced Lie algebra $\mathfrak{g}$, Wildberger constructed its minuscule representations using “minuscule posets”. We want to use this realization of the minuscule representations to study Problem 6. Recall that the weights of a minuscule representation form a finite distributive lattice. Every finite distributive lattice is isomorphic to the lattice of “order filters” on its subposet of “meet irreducible” elements, as we recall below. In this fashion, a minuscule poset distills the lattice of weights of a minuscule representation into a smaller poset. These smaller posets are “d-complete” and are colored by the nodes of the Dynkin diagram so that they become “colored d-complete” posets (see Proctor (1999)).

An element of a lattice is *meet irreducible* if it is covered by exactly one element of the lattice. Let L be a finite distributive lattice, and let P be its sub-poset of meet irreducible elements. An *order filter* of P is a subset $J \subseteq P$ such that if $x \in J$ and $x \preceq y$, then $y \in J$. The lattice L is isomorphic to the lattice $J(P)$ of order filters of P ordered by reverse inclusion. The meet ($\wedge$) and join ($\vee$) operations on the lattice $J(P)$ are the union and intersection operations on filters respectively.

Definition 7 *A minuscule poset P_λ is the subposet of meet irreducible elements in the distributive lattice L_λ of weights which occur in a minuscule representation V_λ.*

The Hasse diagrams of the minuscule posets are displayed in Figure 1. The following notation was established in Proctor (1984): Let $\mathfrak{g}$ be a simple Lie algebra of Dynkin type X_r. Let ω_j be a minuscule weight from Definition 5. The minuscule poset arising from the $\mathfrak{g}$-module V_{ω_j} is denoted $x_r(j)$.

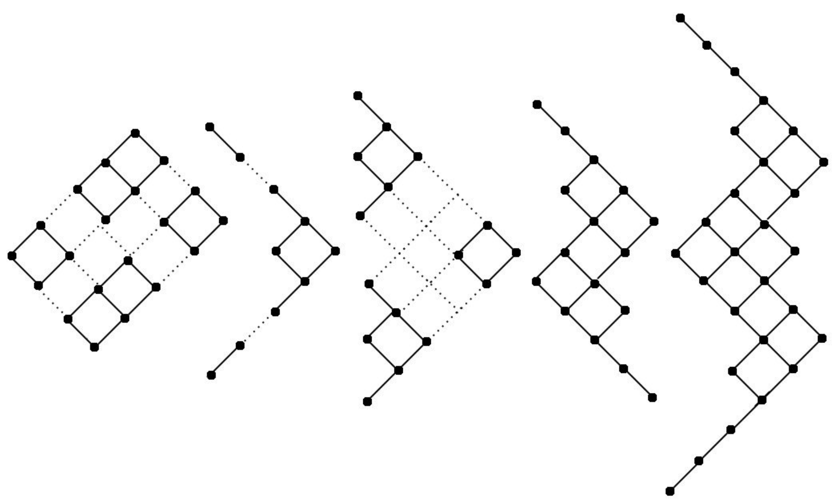

Fig. 1: Hasse diagrams of the minuscule posets $a_n(j)$, $d_n(1)$, $b_n(n) \cong d_n(n-1) \cong d_n(n)$, $e_6(1) \cong e_6(6)$, $e_7(7)$.

Fix a simple Lie algebra $\mathfrak{g}$ and a minuscule weight λ. Use P to denote the minuscule poset $P_\lambda \subseteq L_\lambda$. Let S be the set of nodes of the Dynkin diagram of $\mathfrak{g}$. The minuscule poset P is naturally colored by the function $\kappa : P \to S$ as follows: Let $\mu \in L_\lambda$ be a meet irreducible weight of V_λ. Then μ is covered by exactly one weight $\nu \in L_\lambda$. The difference $\nu - \mu$ is a simple root. The color $\kappa(\mu)$ is the index of this root. The elements of a given color in a minuscule poset form a chain.

The Weyl group acts naturally on the lattice L_λ of weights in V_λ. This action can be described combinatorially with the lattice $J(P)$ of order filters in P. We first need some definitions: Fix a filter $J \subseteq P$ and a color i. There is at most one element $x \in J$ with $\kappa(x) = i$ such that $J \setminus \{x\}$ is a filter. We say that such an element x (which must be minimal in J) and its color i are *removable* from J. Similarly, there is at most one element $y \in P \setminus J$ with $\kappa(y) = i$ such that $J \cup \{y\}$ is a filter. We say that such an the element y (which must be maximal in $P \setminus J$) and its color i are *available* to J. No color is both removable from and available to a given filter.

Proposition 8 *The action of a simple reflection $s_i \in W$ on the weight in $L_\lambda \cong J(P)$ specified by an order filter $J \subseteq P$ is given by the following:*

$$s_i.J = \begin{cases} J \setminus \{x\} & \text{if there exists } x \text{ removable from } J \text{ with } \kappa(x) = i \\ J \cup \{y\} & \text{if there exists } y \text{ available to } J \text{ with } \kappa(y) = i \\ J & \text{otherwise.} \end{cases}$$

Recall that a weight basis for the minuscule representation V_λ is indexed by the elements of its finite distributive lattice L_λ of weights and that this lattice is realized by $J(P)$, which consists of the order filters in the corresponding minuscule poset. Then each vector in a weight basis of V_λ corresponds to a filter in $J(P)$. Wildberger (2003) constructed the minuscule representations for simply laced Lie algebras combinatorially from minuscule posets. We detail his construction below.

From now on, assume that our simple Lie algebra $\mathfrak{g}$ is simply laced. Choose simple root vectors $\{e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}\}_{i \in S}$ where each pair generates a standard $\mathfrak{sl}_2$ subalgebra. Together, these generate $\mathfrak{g}$. We realize the minuscule representation on $V_{J(P)}$, the vector space spanned by linearly independent vectors $\{\mathcal{J} \mid J \subseteq P \text{ an order filter}\}$. When we invoke a filter operation on a vector $\mathcal{J}$ associated to the order filter J, we are indicating the vector associated to the result of the filter operation on J.

Proposition 9 *For each basis vector $\mathcal{J} \in V_{J(P)}$ define the following actions:*

$$e_i.\mathcal{J} := \begin{cases} \mathcal{J} \setminus \{x\} & \text{if there exists } x \text{ removable in } J \text{ with } \kappa(x) = i \\ 0 & \text{otherwise} \end{cases}$$

$$f_i.\mathcal{J} := \begin{cases} \mathcal{J} \cup \{y\} & \text{if there exists } y \text{ available to } J \text{ with } \kappa(y) = i \\ 0 & \text{otherwise.} \end{cases}$$

These actions generate an irreducible representation of $\mathfrak{g}$ on $V_{J(P)}$ that is isomorphic to V_λ. Moreover, each vector $\mathcal{J}$ is a weight vector of $V_{J(P)}$ that has weight $\lambda - \sum_{x \in J} \alpha_{\kappa(x)}$.

There are two special order filters in every poset: the empty filter and the full poset. The vectors in $V_{J(P)}$for these filters are denoted $\aleph$ and $\overline{\aleph}$ respectively. Then $\aleph$ is a highest weight vector for $\mathfrak{g}$ with weight λ. Wildberger chooses a basis of root vectors for $\mathfrak{g}$ and gives their actions on $V_{J(P)}$ using similarly combinatorial formulae. These actions are described by "root layers" of a minuscule poset. Root layers will also appear in our description of Plücker relations:

Definition 10 *For $\alpha \in \Phi^+$, an α-*layer *is a convex subset $R \subseteq P$ with color census $\sum_{x \in R} \alpha_{\kappa(x)} = \alpha$.*

By using Wildberger's realization of V_λ we have the following reformulation of Problem 6:

Problem 11 *The $\mathfrak{g}$-module $Sym^2(V_{J(P)})$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{g}).(\aleph)^2 \oplus I$ for a unique submodule I. A Plücker relation is given by a nonzero vector in I. Find a spanning set (or basis) for I.*

By Seshadri's theorem the submodule I has a basis given by straightening laws, one for each incomparable pair in L_λ. We want to produce the straightening law for as many of these incomparable pairs as possible. If the poset P is a chain, then there are no incomparable pairs. So we will assume that P is not a chain.

4 Model case

The simplest family of minuscule posets which are not chains is the family $d_r(1)$ for $r \geq 3$. These posets are called "double-tailed diamonds" after the shape of their Hasse diagram. Fix such an r and let $\mathfrak{g} = \mathfrak{o}(2r)$. Let λ be the minuscule highest weight of the natural representation of $\mathfrak{g}$. We develop notation for Wildberger's realization of V_λ for this important family of model cases. Label $d_r(1)$'s incomparable pair of elements $z^\sharp$ and $z^\flat$. Then from the middle rank outward, label its tail elements $y^{+/-}, x^{+/-}, \ldots, a^{+/-}$ using $+$ on the upper tail and $-$ on the lower tail. Let $Z^{\sharp/\flat}$ denote the principal filters $\langle z^{\sharp/\flat}\rangle$. Let $Y^+, X^+, \ldots, A^+$ denote the $r-2$ principal filters $\langle y^+\rangle, \langle x^+\rangle, \ldots, \langle a^+\rangle$ contained in the upper tail. Let $Y^-, X^-, \ldots, A^-$ denote the $r-2$ filters $\langle y^-\rangle \setminus y^-, \langle x^-\rangle \setminus x^-, \ldots, \langle a^-\rangle \setminus a^-$. The remaining two filters are $\emptyset$ and $P = d_r(1)$ itself. There is a unique Plücker relation in this case (up to scalar multiple):

Proposition 12 *The submodule $I \subset Sym^2(V_{J(P)})$ is spanned by the alternating sum:*

$$\mathcal{Z}^\sharp\mathcal{Z}^\flat - \mathcal{Y}^+\mathcal{Y}^- + \mathcal{X}^+\mathcal{X}^- - \cdots + (-1)^{r-2}\mathcal{A}^+\mathcal{A}^- + (-1)^{r-1}\aleph\overline{\aleph}.$$

Hence the following quadratic Plücker relation has been obtained:

$$\mathcal{Z}^\sharp\mathcal{Z}^\flat = \mathcal{Y}^+\mathcal{Y}^- - \mathcal{X}^+\mathcal{X}^- + \cdots + (-1)^{r-1}\mathcal{A}^+\mathcal{A}^- + (-1)^{r}\aleph\overline{\aleph}.$$

This Plücker relation is a straightening law, so this proposition verifies Seshadri's theorem here. Also note that $\mathcal{Z}^\sharp \wedge \mathcal{Z}^\flat = \mathcal{Y}^-$ and $\mathcal{Z}^\sharp \vee \mathcal{Z}^\flat = \mathcal{Y}^+$, so that the leading term on the right hand side of the straightening law is the product of the meet and join of the incomparable pair. There are as many terms in this expression as the rank r of the our algebra. Lastly, note that the last sign in the above relation depends on the parity of r. This is a common phenomenon for type D_r objects.

5 Highest weight relation

First, we obtain a highest weight vector of I for $\mathfrak{g}$. In the model cases $P = d_n(1)$ or the small cases $a_3(2) \cong d_3(1)$ and $d_4(3) \cong d_4(4) \cong d_4(1)$, this vector alone forms a basis. The highest weight vector we find will give one straightening law for the corresponding coordinate ring. So in these cases we also have an explicit presentation of the straightening law for every (the only) incomparable pair.

Let $P := P_\lambda \subset L_\lambda$ be a minuscule poset that is not a chain. Notice that at the top of P there is a double-tailed diamond subposet P_D, *i.e.* an order filter that is isomorphic to the minuscule poset $d_r(1)$ for some $r \leq n$. For the type D model case of $P = d_n(1)$, we have $r = n$ and this top double-tailed diamond is all of P. For type A we have $r = 3$, for the type D spin representations we have $r = 4$, for type E_6 we have $r = 5$, and for type E_7 we have $r = 6$. Forget the usual indexing of simple roots. Use $\alpha_\sharp$ and $\alpha_\flat$ to denote the simple roots of Φ which correspond to the colors of the incomparable pair of $P_D \subseteq P$. Use $\alpha_1, \ldots, \alpha_{r-2}$ to denote the simple roots of Φ which correspond to the colors of the elements of P_D

above the incomparable pair, numbering upward. Let $\Phi_D \subseteq \Phi$ be the subset of roots in the span of $\alpha_\sharp, \alpha_\flat, \alpha_1, \ldots, \alpha_{r-2}$. Let $\mathfrak{g}_D \subseteq \mathfrak{g}$ be the subalgebra generated by the root subspaces $\{\mathfrak{g}_\alpha\}_{\alpha \in \Phi_D}$. The subalgebra $\mathfrak{g}_D$ is simple of type D_r.

Any $\mathfrak{g}$-module is naturally a $\mathfrak{g}_D$-module, which can be decomposed into $\mathfrak{g}_D$-irreducible components. Let V_D denote the $\mathfrak{g}_D$-irreducible component of $V_{J(P)}$ given by $\mathfrak{U}(\mathfrak{g}_D).\aleph$; it is isomorphic to $V_{J(d_r(1))}$ as a $\mathfrak{g}_D$-module. Use the notation established in Section 4 for $V_{J(d_r(1))}$ here for V_D, but use $\aleph_D^-$ to denote the full double-tailed diamond filter there. Problem 11 for the $\mathfrak{g}_D$-submodule V_D was solved in Proposition 12; there is a single Plücker relation for $\mathfrak{g}_D$ in $Sym^2(V_D)$:

$$\mathcal{Z}^\sharp \mathcal{Z}^\flat = \mathcal{Y}^+\mathcal{Y}^- - \mathcal{X}^+\mathcal{X}^- + \cdots + (-1)^{r-1}\mathcal{A}^+\mathcal{A}^- + (-1)^r \aleph\aleph_D^-.$$

The inclusion $V_D \subseteq V_{J(P)}$ induces a natural inclusion $Sym^2(V_D) \hookrightarrow Sym^2(V_{J(P)})$. Under this inclusion the model relation above is the foremost of the Plücker relations we seek:

Proposition 13 *The inclusion of the above Plücker relation for $\mathfrak{g}_D$ under $Sym^2(V_D) \hookrightarrow Sym^2(V_{J(P)})$ is a Plücker relation for $\mathfrak{g}$. Moreover, this relation is a highest weight vector for $\mathfrak{g}$.*

Proof: These Plücker relations are calculated using Casimir operators. The computation for "the highest" relation for $\mathfrak{g}$ reduces to the computation for the above Plücker relation for the subalgebra $\mathfrak{g}_D$. The reduction uses weight calculations. $\square$

6 Rotation by Weyl group

Let η denote the weight of the Plücker relation from Proposition 13. Seshadri's theorem can be used to see that every highest weight of I is dominated by η. Hence η is the unique maximal weight of I. The weights in the Weyl group orbit of η are also weights of I (and are weights only for the same irreducible component as the weight η relation above). We call these *extreme weights*.

Definition 14 *An* extreme weight Plücker relation *is a nonzero weight vector of I that has weight $w.\eta$ for some $w \in W$.*

When combined with the action of the Weyl group, the technique used in Proposition 13 to find a highest weight Plücker relation can be used to produce extreme weight Plücker relations. We describe the $|W.\eta|$ extreme weight vectors for I in terms of the order structure of the elements of its "support" in L_λ. In particular, we will see directly that these extreme weight relations are straightening laws.

Definition 15 *A* double-tailed diamond subposet *of a poset P is a subset $P_D \subseteq P$ such that there are exactly two incomparable elements of P_D, half of the other elements of P_D form a chain that lies above the incomparable pair, and the remaining elements form a chain that lies below the incomparable pair.*

A double-tailed diamond sublattice *of a lattice L is a double-tailed diamond subposet $L_D \subseteq L$ such that the join of its incomparable pair in L is the minimal element in the upper chain of L_D, and the meet of its incomparable pair in L is the maximal element in the lower chain of L_D.*

Suppose that L generates a ring over $\mathbb{C}$. A double-tailed diamond sublattice $L_D \subseteq L$ is order isomorphic to one of the model lattices $J(d_r(1))$. In this ring, we call a relation of the form obtained in Proposition 12 the *standard straightening law* on L_D. The following theorem is our foremost result:

Theorem 16 *Let λ be a minuscule weight of a simple Lie algebra $\mathfrak{g}$. Each extreme weight Plücker relation is the standard straightening law on a double-tailed diamond sublattice of L_λ.*

So an extreme weight Plücker relation is a straightening law for an "extreme" incomparable pair of L_λ. The standard monomial expansion of such a pair begins with the product of their meet and join, and continues as an alternating sum of products of comparable elements in a double-tailed diamond. These double-tailed diamonds will all be the same size as the one found in the previous section. Hence there are again r terms in the straightening law including the incomparable pair, where r is the rank of the subalgebra $\mathfrak{g}_D$ defined there. For now, continue to assume that $\mathfrak{g}$ is simply laced. Here this theorem is obtained by combining Propositions 17 and 18 below. In addition, Corollary 19 below indicates that the difference between the filters for two adjacent elements of this sublattice is a root layer. Section 7 extends the theorem to non-simply laced algebras.

Let $W_\eta \subseteq W$ be the parabolic subgroup which stabilizes the weight η. Each coset in W/W_η is known to have a shortest length representative. Let W^η be the set of such representatives. The set of extreme weights is in bijection with W^η. Since W^η is defined in terms of coset representatives, it is a subset of the Weyl group. We will apply elements of W^η in settings where η is not a highest weight.

Fix an element $w \in W^\eta$. We now "rotate" the setup for Proposition 13 using w. Recall the root subsystem $\Phi_D \subseteq \Phi$ defined there. Let $\mathfrak{g}_{D,w} \subseteq \mathfrak{g}$ be the subalgebra generated by the root subspaces $\{\mathfrak{g}_\alpha\}_{\alpha \in w.\Phi_D}$. Again any $\mathfrak{g}$-module is naturally a $\mathfrak{g}_{D,w}$-module, which can be decomposed into $\mathfrak{g}_{D,w}$-irreducible components. Let $V_{D,w}$ denote the $\mathfrak{g}_{D,w}$-irreducible component of $V_{J(P)}$ given by $\mathfrak{U}(\mathfrak{g}_D).(w.\aleph)$. It is straightforward to apply the techniques of Proposition 13 to $V_{D,w} \subseteq V_{J(P)}$:

Proposition 17 *Let $w \in W^\eta$. The inclusion of the Plücker relation for $\mathfrak{g}_{D,w}$ under $Sym^2(V_{D,w}) \hookrightarrow Sym^2(V_{J(P)})$ is an extreme weight Plücker relation for $\mathfrak{g}$ of weight $w.\eta$.*

From this proposition we obtain an extreme weight relation which is an alternating sum of products of pairs of filters as in Proposition 12. We want to understand the order structure of these filters to prove that it is a standard straightening law on a double-tailed diamond sublattice of L_λ. The lattice L_D of weights of V_D is a sublattice of L_λ. The corresponding extreme Plücker relation at $w = id$ of Proposition 13 was then the standard straightening law on this sublattice. Let $L_{D,w}$ denote the lattice of weights of $V_{D,w}$; again we have $L_{D,w} \subseteq L_\lambda$. For an arbitrary $w \in W^\eta$ the lattice $L_{D,w}$ has the same order properties as the sublattice L_D:

Proposition 18 *Let $\mu, \nu \in L_D$, and let $w \in W^\eta$. Then $w.\mu \preceq w.\nu$ if and only if $\mu \preceq \nu$. Hence $L_{D,w} = w.L_D$ is order isomorphic to L_D. Moreover, it is a double-tailed diamond sublattice of L_λ.*

Proof: This is proved by induction on the length of w. □

This proposition implies that the extreme weight Plücker relation obtained by Proposition 17 is the standard straightening law on $L_{D,w}$. The proof of Theorem 16 is now complete for simply laced algebras $\mathfrak{g}$. An immediate corollary to the proof of the previous proposition describes the covering relations of $L_{D,w}$, using the language of filters:

Corollary 19 *Let $w \in W^\eta$. Let J, K be filters of the minuscule poset P such that J covers K in the sublattice $L_{D,w} \subseteq L_\lambda \cong J(P)$. Then there exists a root $\alpha \in w.\Phi_D^+ \subseteq \Phi^+$ such that the subset $K - J$ is an α-layer. If one of J or K is one of the incomparable pair of elements of $L_{D,w}$, then $\alpha = w.\alpha_\sharp$ or $\alpha = w.\alpha_\flat$. Otherwise, moving outward along the tails from the incomparable pair, the root α takes on the $r - 2$ values $w.\alpha_1, \ldots, w.\alpha_{r-2}$.*

7 Exceptional cases

If I itself is also a minuscule $\mathfrak{g}$-module, then every weight of I is an extreme weight. In this case, the extreme weight Plücker relations form a basis of all of the Plücker relations. Here we are presenting the straightening law for every incomparable pair in L_λ. This is the case for $P = e_6(1)$ and $P = e_6(6)$. (It can be seen that I is also minuscule for $P = a_n(2)$ and $P = a_n(n-1)$.) A "quasiminuscule" representation is an irreducible representation in which every nonzero weight lies in the Weyl group orbit of its highest weight. If I is quasiminuscule, then the extreme weight Plücker relations give the straightening law for all but the zero weight incomparable pairs in L_λ. This is the case for the remaining exceptional example $P = e_7(7)$. Here the zero weight space of I is seven dimensional. A basis for the zero weight space can be computed by hand. In these cases we solve Problem 11 completely:

Theorem 20 *For the two type E_6 cases, the Plücker relations described by Theorem 16 form a basis of Plücker relations. For the type E_7 case, they combine with the seven relations of zero weight in Figure 4 to form a basis of Plücker relations.*

We obtain the straightening law for every incomparable pair in L_λ for these exceptional cases. In type E_6, Theorem 16 described the straightening laws for every incomparable pair. We list these 27 straightening laws in Figure 3. In type E_7, Theorem 16 described the straightening laws for 126 of the 133 incomparable pairs. We display the 7 remaining weight zero straightening laws in Figure 4. Unlike the extreme weight straightening laws, these 7 zero weight straightening laws were computed by hand. This involved Gaussian elimination to solve for the straightening laws using another basis for the zero weight space of I.

We establish some notation for these cases. Label the Dynkin diagram of E_6 with letters $\{a, b, c, d, e, o\}$ and the diagram of E_7 with letters $\{a, b, c, d, e, f, o\}$. This leads to the coloring of the minuscule posets shown in Figure 2. Recall that elements of a minuscule poset with a given color form a chain. We will name an element with its color and a subscript that indicates its position in this chain, counting from the top. We name a filter by the capitalized string of its minimal elements. For example, in both posets, the filter A_2 is the top double-tailed diamond. In $e_7(7)$, the filter A_2E_2 also includes the elements f_1 and e_2. We keep our usual convention of using calligraphic font to name a basis vector of Wildberger's $\mathfrak{g}$-module $V_{J(P)}$. Since we are using strings of letters to name filters, we place a dot between two vectors of $V_{J(P)}$ to indicate their product in $Sym^2(V_{J(P)})$.

Since we are listing only the zero weight relations in the E_7 case, every filter will appear paired with the filter corresponding to the negative of its weight. We denote by $\overline{J}$ the pair of the filter J. For example the principal filter genereated by f_2 is named $\overline{A_2}$, since the weight of the vector $\mathcal{A}_2 \cdot \overline{\mathcal{A}_2}$ is zero. There are 28 zero weight pairs of filters. Seven of these are the incomparable pairs, while the other 21 are the standard monomials. Figure 4 displays a matrix which lists the 21 standard monomial cooordinates for each of the 7 products of incomparable pairs. For legibility, negative coordinates are presented with bars. To display the relations in matrix form, we must fix some total ordering of the monomials. We use the following arbitrary order for the products of incomparable pairs:

$$\mathcal{A}_2 \cdot \mathcal{F}_2, \quad \mathcal{E}_3 \cdot \mathcal{A}_2\mathcal{F}_1, \quad \mathcal{D}_4 \cdot \mathcal{A}_2\mathcal{E}_2, \quad \mathcal{A}_2\mathcal{D}_3 \cdot \mathcal{C}_3\mathcal{O}_2, \quad \mathcal{B}_2\mathcal{O}_2 \cdot \mathcal{A}_2\mathcal{C}_3, \quad \mathcal{B}_3 \cdot \mathcal{O}_2, \quad \mathcal{C}_3 \cdot \mathcal{A}_2\mathcal{O}_2$$

We order the standard monomials by the following reverse lexicographic order:

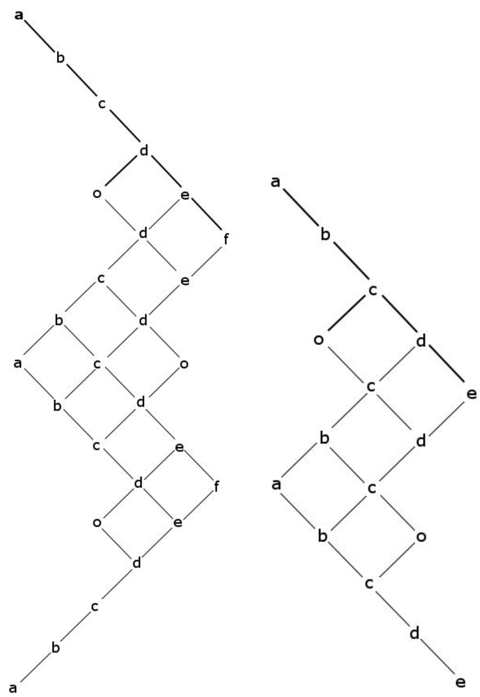

Fig. 2: The colored Hasse diagrams for $e_7(7)$ and $e_6(1)$ (or $e_6(6)$).

$$\mathcal{B}_2\mathcal{D}_3 \cdot \overline{\mathcal{B}_2\mathcal{D}_3},\ \mathcal{B}_2\mathcal{E}_2 \cdot \overline{\mathcal{B}_2\mathcal{E}_2},\ \mathcal{B}_2\mathcal{F}_1 \cdot \overline{\mathcal{B}_2\mathcal{F}_1},\ \mathcal{B}_2 \cdot \overline{\mathcal{B}_2},\ \mathcal{D}_3 \cdot \overline{\mathcal{D}_3},\ \mathcal{C}_2\mathcal{E}_2 \cdot \overline{\mathcal{C}_2\mathcal{E}_2},\ \mathcal{C}_2\mathcal{F}_1 \cdot \overline{\mathcal{C}_2\mathcal{F}_1},$$
$$\mathcal{C}_2 \cdot \overline{\mathcal{C}_2},\ \mathcal{E}_2 \cdot \overline{\mathcal{E}_2},\ \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1},\ \mathcal{D}_2 \cdot \overline{\mathcal{D}_2},\ \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1},\ \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1},\ \mathcal{O}_1 \cdot \overline{\mathcal{O}_1},$$
$$\mathcal{F}_1 \cdot \overline{\mathcal{F}_1},\ \mathcal{E}_1 \cdot \overline{\mathcal{E}_1},\ \mathcal{D}_1 \cdot \overline{\mathcal{D}_1},\ \mathcal{C}_1 \cdot \overline{\mathcal{C}_1},\ \mathcal{B}_1 \cdot \overline{\mathcal{B}_1},\ \mathcal{A}_1 \cdot \overline{\mathcal{A}_1},\ \aleph \cdot \overline{\aleph}$$

8 Non-simply laced cases

Beginning with Section 5, we assumed that our algebra $\mathfrak{g}$ was simply laced. However, there are minuscule weights in the non-simply laced type B and C root systems. The weight ω_1 of the type C_n system is minuscule. There a dimension calculation shows that the corresponding Plücker module $I = 0$, so there are no Plücker relations in this case. The weight ω_n of the type B_n system is also minuscule. For this case we deduce results about its Plücker relations from our results for a simply laced type D case through the strategy of "diagram folding." This uses an embedding of a type B_n Lie algebra into one of type D_{n+1}.

Fix $n \geq 2$. Let $\mathfrak{g}$ be a simple Lie algebra of type D_{n+1}. There is an embedded type B_n subalgebra $\mathfrak{g}_B \subset \mathfrak{g}$; it can be defined as the fixed points of an automorphism of $\mathfrak{g}$. The subalgebra $\mathfrak{h}_B := \mathfrak{h} \cap \mathfrak{g}_B$ is a Cartan subalgebra for $\mathfrak{g}_B$. The minuscule weight λ_B of $\mathfrak{h}_B^*$ is the restriction $\omega_n|_{\mathfrak{h}_B}$ (or $\omega_{n+1}|_{\mathfrak{h}_B}$) of a spin minuscule weight of $\mathfrak{h}^*$. Let λ be one of these two spin weights of $\mathfrak{g}$. Let $P := P_\lambda$ be the corresponding minuscule poset, and construct Wildberger's representation $V_{J(P)}$ of $\mathfrak{g}$. Define $I \subset Sym^2(V_{J(P)})$ as for Problem 11. The subalgebra $\mathfrak{g}_B \subset \mathfrak{g}$ acts naturally on $V_{J(P)}$. Then the $\mathfrak{g}_B$ submodule $V_B := \mathfrak{U}(\mathfrak{g}_B).\aleph$ of $V_{J(P)}$ is a minuscule representation of $\mathfrak{g}_B$ with highest weight λ_B. One can pose Problem 11 for

Inc. Pair		Meet · Join						
$\mathcal{D}_1 \cdot \mathcal{O}_1$	=	$\mathcal{C}_1 \cdot \mathcal{D}_1\mathcal{O}_1$	-	$\mathcal{B}_1 \cdot \mathcal{C}_2$	+	$\mathcal{A}_1 \cdot \mathcal{B}_2$	-	$\aleph \cdot \mathcal{A}_2$
$\mathcal{E}_1 \cdot \mathcal{O}_1$	=	$\mathcal{C}_1 \cdot \mathcal{E}_1\mathcal{O}_1$	-	$\mathcal{B}_1 \cdot \mathcal{C}_2\mathcal{E}_1$	+	$\mathcal{A}_1 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\aleph \cdot \mathcal{A}_2\mathcal{E}_1$
$\mathcal{E}_1 \cdot \mathcal{D}_1\mathcal{O}_1$	=	$\mathcal{D}_1 \cdot \mathcal{E}_1\mathcal{O}_1$	-	$\mathcal{B}_1 \cdot \mathcal{D}_2$	+	$\mathcal{A}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	-	$\aleph \cdot \mathcal{A}_2\mathcal{D}_2$
$\mathcal{C}_2 \cdot \mathcal{E}_1$	=	$\mathcal{D}_1 \cdot \mathcal{C}_2\mathcal{E}_1$	-	$\mathcal{C}_1 \cdot \mathcal{D}_2$	+	$\mathcal{A}_1 \cdot \mathcal{C}_3$	-	$\aleph \cdot \mathcal{A}_2\mathcal{C}_3$
$\mathcal{C}_2 \cdot \mathcal{E}_1\mathcal{O}_1$	=	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{C}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{D}_2$	+	$\mathcal{A}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{A}_2\mathcal{O}_2$
$\mathcal{B}_2 \cdot \mathcal{E}_1$	=	$\mathcal{D}_1 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\mathcal{C}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{C}_3$	-	$\aleph \cdot \mathcal{B}_3$
$\mathcal{B}_2 \cdot \mathcal{E}_1\mathcal{O}_1$	=	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{B}_3\mathcal{O}_2$
$\mathcal{A}_2 \cdot \mathcal{E}_1$	=	$\mathcal{D}_1 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{C}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{A}_1 \cdot \mathcal{B}_3$
$\mathcal{B}_2 \cdot \mathcal{C}_2\mathcal{E}_1$	=	$\mathcal{C}_2 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{C}_3$	+	$\mathcal{C}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{C}_4$
$\mathcal{A}_2 \cdot \mathcal{E}_1\mathcal{O}_1$	=	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{B}_3\mathcal{O}_2$
$\mathcal{B}_2 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2 \cdot \mathcal{B}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{D}_1 \cdot \mathcal{C}_3$	+	$\mathcal{D}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{D}_3$
$\mathcal{A}_2 \cdot \mathcal{C}_2\mathcal{E}_1$	=	$\mathcal{C}_2 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	+	$\mathcal{C}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{C}_4$
$\mathcal{A}_2 \cdot \mathcal{B}_2\mathcal{E}_1$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{B}_3$	+	$\mathcal{C}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_1 \cdot \mathcal{C}_4$
$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{E}_1 \cdot \mathcal{C}_3$	+	$\mathcal{E}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \overline{\aleph}$
$\mathcal{A}_2 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{D}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	+	$\mathcal{D}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2 \cdot \mathcal{B}_2\mathcal{D}_2$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{D}_1 \cdot \mathcal{B}_3$	+	$\mathcal{D}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	+	$\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \overline{\aleph}$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	=	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{E}_1 \cdot \mathcal{B}_3$	+	$\mathcal{E}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_1 \cdot \overline{\aleph}$
$\mathcal{A}_2 \cdot \mathcal{C}_3$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{C}_2 \cdot \mathcal{B}_3$	+	$\mathcal{D}_1 \cdot \mathcal{C}_4$	-	$\mathcal{C}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2 \cdot \mathcal{O}_2$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{C}_2 \cdot \mathcal{B}_3\mathcal{O}_2$	+	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{C}_4$	-	$\mathcal{O}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{C}_3$	=	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{B}_3$	+	$\mathcal{E}_1 \cdot \mathcal{C}_4$	-	$\mathcal{C}_1 \cdot \overline{\aleph}$
$\mathcal{A}_2\mathcal{D}_2 \cdot \mathcal{C}_3$	=	$\mathcal{B}_2\mathcal{D}_2 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{D}_2 \cdot \mathcal{B}_3$	+	$\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{D}_1 \cdot \overline{\aleph}$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{O}_2$	=	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	+	$\mathcal{E}_1\mathcal{O}_1 \cdot \mathcal{C}_4$	-	$\mathcal{O}_1 \cdot \overline{\aleph}$
$\mathcal{A}_2\mathcal{D}_2 \cdot \mathcal{O}_2$	=	$\mathcal{B}_2\mathcal{D}_2 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{D}_2 \cdot \mathcal{B}_3\mathcal{O}_2$	+	$\mathcal{E}_1\mathcal{O}_1 \cdot \mathcal{D}_3$	-	$\mathcal{D}_1\mathcal{O}_1 \cdot \overline{\aleph}$
$\mathcal{A}_2\mathcal{C}_3 \cdot \mathcal{O}_2$	=	$\mathcal{C}_3 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{D}_2 \cdot \mathcal{C}_4$	+	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{C}_2 \cdot \overline{\aleph}$
$\mathcal{B}_3 \cdot \mathcal{O}_2$	=	$\mathcal{C}_3 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_2\mathcal{D}_2 \cdot \mathcal{C}_4$	+	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{B}_2 \cdot \overline{\aleph}$
$\mathcal{B}_3 \cdot \mathcal{A}_2\mathcal{O}_2$	=	$\mathcal{A}_2\mathcal{C}_3 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{A}_2\mathcal{D}_2 \cdot \mathcal{C}_4$	+	$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{A}_2 \cdot \overline{\aleph}$

Fig. 3: The 27 straightening laws for the complex Cayley plane on its Plücker coordinates.

the $\mathfrak{g}_B$-module V_B: The $\mathfrak{g}_B$-module $Sym^2(V_B)$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{g}_B).(\aleph)^2 \oplus I_B$ of $\mathfrak{g}_B$-submodules for some submodule I_B. Find a spanning set (or basis) for I_B.

Proposition 21 *The $\mathfrak{g}_B$-modules V_B and $V_{J(P)}$ are equal. Moreover, the subspaces I and I_B of $Sym^2(V_{J(P)}) = Sym^2(V_B)$ are equal.*

This proposition allows us to obtain some Plücker relations for $\mathfrak{g}_B$ by first applying Section 6 with the simply laced algebra $\mathfrak{g}$ to obtain extreme weight relations in I, and then recognizing those as Plücker relations in I_B. We know these are standard straightening laws on double-tailed diamond sublattices of L_λ. By checking their order structure in L_{λ_B}, we obtain the final ingredient for Theorem 16:

Corollary 22 *Each extreme weight Plücker relation for $\mathfrak{g}_B$ is the standard straightening law on a double-tailed diamond sublattice of L_{λ_B}. Moreover, Corollary 19 also holds here.*

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$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} \\
0 & 0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & \bar{1} & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 & \bar{1} & 1 \\
0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 & 1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 & \bar{1} \\
1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & 0 & 0 & 0 & 1 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 2 & \bar{2} \\
1 & \bar{1} & 1 & \bar{1} & 0 & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 2 & \bar{1} & 2 \\
1 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 3
\end{bmatrix}$$

Fig. 4: The matrix which lists the 21 standard monomial coordinates of the 7 products of weight zero incomparable pairs of Plücker coordinates for the Freudenthal variety, in the total order of Section 7.

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