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The facial weak order in finite Coxeter groups

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Abstract. We investigate a poset structure that extends the weak order on a finite Coxeter group W to the set of all faces of the permutahedron of W. We call this order the *facial weak order*. We first provide two alternative characterizations of this poset: a first one, geometric, that generalizes the notion of inversion sets of roots, and a second one, combinatorial, that uses comparisons of the minimal and maximal length representatives of the cosets. These characterizations are then used to show that the facial weak order is in fact a lattice, generalizing a well-known result of A. Björner for the classical weak order. Finally, we show that any lattice congruence of the classical weak order induces a lattice congruence of the facial weak order, and we give a geometric interpretation of its classes.

Résumé. Nous étudions une structure de poset qui étend l'ordre faible sur un groupe de Coxeter fini W à l'ensemble de toutes les faces du permutoèdre de W. Nous appellons cet ordre *l'ordre faible facial*. Nous montrons d'abord deux caractérisations alternatives de ce poset : une première, géométrique, qui généralise la notion d'ensemble d'inversion, et une seconde, combinatoire, qui utilise des comparaisons entre les représentants minimaux et maximaux des faces. Ces caractérisations sont ensuite utilisées pour montrer que l'ordre faible facial est en fait un treillis, généralisant ainsi un résultat bien connu de A. Björner pour l'ordre faible classique. Finalement, nous montrons que toute congruence de treillis de l'ordre faible classique induit une congruence de treillis de l'ordre faible facial, et nous donnons une interprétation géométrique de ses classes.

Keywords. Permutahedra, weak order, Coxeter complex

The (right) Cayley graph of a Coxeter system (W, S) is naturally oriented by the (right) weak order on W: an edge is oriented from w to ws if $s \in S$ is such that $\ell(w) < \ell(ws)$, see [BB05, Chapter 3] for details. A celebrated result of A. Björner [Bjö84] states that the weak order is a complete meet-semilattice and even a complete ortholattice in the case of a finite Coxeter system. The weak order is a very useful tool to study Coxeter groups as it encodes the combinatorics of reduced words associated to (W, S), and it underlines the connection between the words and the root system via the notion of inversion sets, see for instance [Dye11, HL16] and the references therein.

In the case of a finite Coxeter system, the Cayley graph of W is isomorphic to the 1-skeleton of the W-permutahedron. Then the weak order is given by an orientation of the 1-skeleton of the W-permutahedron associated to the choice of a linear form of the ambiant Euclidean space. This point of view

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was very useful in order to build generalized associahedra out of a W-permutahedron using N. Reading's Cambrian lattices, see [Rea12, HLT11, Hoh12].

We study a poset structure on all faces of the W-permutahedron that we call the *facial weak order*. This order was introduced by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer in [KLN$^+$01] for the symmetric group then extended by P. Palacios and M. Ronco in [PR06] for arbitrary finite Coxeter groups. Recall that the faces of the W-permutahedron are naturally parameterized by the Coxeter complex $\mathcal{P}_W$ which consists of all standard parabolic cosets W/W_I for $I \subseteq S$. The aims of this article are:

1. To give two alternative characterizations of the facial weak order (see Theorem 10): one in terms of root inversion sets of parabolic cosets which extend the notion of inversion sets of elements of W, and the other using weak order comparisons between the minimal and maximal representatives of the parabolic cosets. The advantage of these two definitions is that they give immediate global comparison, while the original definition of [PR06] uses cover relations.
2. To show that the facial weak order is a lattice (see Theorem 13), whose restriction to the vertices of the permutahedron produces the weak order as a sublattice. This result was motivated by the special case of type A proved in [KLN$^+$01].
3. To show that any lattice congruence $\equiv$ of the weak order extends to a lattice congruence $\equiv_{\mathcal{P}}$ of the facial weak order (see Theorem 18). This provides a complete description (see Theorem 21) of all cones of the simplicial fan $\mathcal{F}_{\equiv}$ associated to the weak order congruence $\equiv$ in [Rea05].

The results of this paper are based on combinatorial properties of Coxeter groups, parabolic cosets, and reduced words. However, their motivation and intuition come from the geometry of the Coxeter arrangement and of the W-permutahedron. So we made a point to introduce enough of the geometrical material to make the geometric intuition clear.

1 Preliminaries

We start by fixing notations and classical definitions on finite Coxeter groups. Details can be found in textbooks by J. Humphreys [Hum90] and A. Björner and F. Brenti [BB05]. The reader familiar with finite Coxeter groups and root systems is invited to proceed directly to Section 2.

1.1 Finite reflection groups and Coxeter systems

Let $(V, \langle \cdot \mid \cdot \rangle)$ be an n-dimensional Euclidean vector space. For any vector $v \in V \smallsetminus \{0\}$, we denote by s_v the reflection interchanging v and $-v$ while fixing the orthogonal hyperplane pointwise. Remember that $ws_v = s_{w(v)}w$ for any vector $v \in V \smallsetminus \{0\}$ and any orthogonal transformation w of V.

We consider a *finite reflection group* W acting on V, that is, a finite group generated by reflections in the orthogonal group $O(V)$. The *Coxeter arrangement* of W is the collection of all reflecting hyperplanes. Its complement in V is a union of open polyhedral cones. Their closures are called *chambers*. The *Coxeter fan* is the polyhedral fan formed by the chambers together with all their faces. This fan is *complete* (its cones cover V) and *simplicial* (all cones are simplicial), and we can assume without loss of generality that it is *essential* (the intersection of all chambers is reduced to the origin). We fix an arbitrary chamber $\mathcal{C}$, called *fundamental chamber*. The n reflections orthogonal to the facet defining hyperplanes of $\mathcal{C}$ are called *simple reflections*. The set S of simple reflections generates W. The pair (W, S) forms a *Coxeter system*.

1.2 Roots and weights

We consider a *root system* Φ for W, *i.e.,* a set of vectors invariant under the action of W and containing precisely two opposite roots orthogonal to each reflecting hyperplane of W. The *simple roots* Δ are the roots orthogonal to the defining hyperplanes of $\mathcal{C}$ and pointing towards $\mathcal{C}$. They form a linear basis of V. The root system Φ splits into the *positive roots* $\Phi^+ := \Phi \cap \operatorname{cone}(\Delta)$ and the *negative roots* $\Phi^- := \Phi \cap \operatorname{cone}(-\Delta) = -\Phi^+$, where $\operatorname{cone}(X)$ denotes the set of nonnegative linear combinations of vectors in $X \subseteq V$. In other words, the positive roots are the roots whose scalar product with any vector of the interior of the fundamental chamber $\mathcal{C}$ is positive, and the simple roots form the basis of the cone generated by Φ^+. Each reflection hyperplane is orthogonal to one positive and one negative root. For a reflection $s \in R$, we set α_s to be the unique positive root orthogonal to the reflection hyperplane of s.

We denote by $\alpha_s^\vee := 2\alpha_s / \langle \alpha_s \mid \alpha_s \rangle$ the *coroot* corresponding to $\alpha_s \in \Delta$, and by $\Delta^\vee := \{\alpha_s^\vee \mid s \in S\}$ the coroot basis. The vectors of its dual basis $\nabla := \{\omega_s \mid s \in S\}$ are called *fundamental weights*. In other words, the fundamental weights of W are defined by $\langle \alpha_s^\vee \mid \omega_t \rangle = \delta_{s=t}$ for all $s, t \in S$. Geometrically, the fundamental weight ω_s gives the direction of the ray of the fundamental chamber $\mathcal{C}$ not contained in the reflecting hyperplane of s. We let $\Omega := W(\nabla) = \{w(\omega_s) \mid w \in W, s \in S\}$ denote the set of all weights of W, obtained as the orbit of the fundamental weights under W.

1.3 Length, reduced words and weak order

The *length* $\ell(w)$ of an element $w \in W$ is the length of the smallest word for w as a product of generators in S. A word $w = s_1 \cdots s_k$ with $s_1, \dots, s_k \in S$ is called *reduced* if $k = \ell(w)$. For $u, v \in W$, the product uv is said to be *reduced* if the concatenation of a reduced word for u and of a reduced word for v is a reduced word for uv, *i.e.,* if $\ell(uv) = \ell(u) + \ell(v)$. We say that $u \in W$ is a *prefix* of $v \in W$ if there is a reduced word for u that is the prefix of a reduced word for v, *i.e.,* if $\ell(u^{-1}v) = \ell(v) - \ell(u)$.

The (right) *weak order* is the order on W defined equivalently by

$$u \le v \iff \ell(u) + \ell(u^{-1}v) = \ell(v) \iff u \text{ is a prefix of } v.$$

A. Björner shows in [Bjö84] that the weak order defines a lattice structure on W (finite Coxeter group), with minimal element e and maximal element $w_\circ$ (which sends all positive roots to negative ones and all positive simple roots to negative simple ones). The conjugation $w \mapsto w_\circ w w_\circ$ defines a weak order automorphism while the left and right multiplications $w \mapsto w_\circ w$ and $w \mapsto w w_\circ$ define weak order anti-automorphisms. We refer the reader to [BB05, Chapter 3] for more details.

The weak order encodes the combinatorics of reduced words and enjoys a useful geometric characterization within the root system, which we explain now. The (left) *inversion set* of w is the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$ of positive roots sent to negative ones by w^{-1}. If $w = uv$ is reduced then $\mathbf{N}(w) = \mathbf{N}(u) \sqcup u\big(\mathbf{N}(v)\big)$. In particular, we have $\mathbf{N}(w) = \big\{\alpha_{s_1}, s_1(\alpha_{s_2}), \dots, s_1 s_2 \cdots s_{p-1}(\alpha_{s_k})\big\}$ for any reduced word $w = s_1 \cdots s_k$, and therefore $\ell(w) = |\mathbf{N}(w)|$. Moreover, the weak order is characterized in term of inversion sets by:

$$u \le v \iff \mathbf{N}(u) \subseteq \mathbf{N}(v),$$

for any $u, v \in W$. We refer for instance the reader to [HL16, Section 2] and the references therein.

We say that a simple reflection $s \in S$ is a *left ascent* of $w \in W$ if $\ell(sw) = \ell(w) + 1$ and a *left descent* of w if $\ell(sw) = \ell(w) - 1$. We denote by $D_L(w)$ the set of left descents of w. Note that for $s \in S$ and $w \in W$, we have $s \in D_L(w) \iff \alpha_s \in \mathbf{N}(w) \iff s \le w$. Similarly, $s \in S$ is a *right descent* of $w \in W$ if $\ell(ws) = \ell(w) - 1$, and we denote by $D_R(w)$ the set of right descents of w.

1.4 Parabolic subgroups and cosets

The *standard parabolic subgroup* W_I is the subgroup of W generated by $I \subseteq S$. It is also a Coxeter group with simple generators I, simple roots $\Delta_I := \{\alpha_s \mid s \in I\}$, root system $\Phi_I = W_I(\Delta_I) = \Phi \cap \mathrm{vect}(\Delta_I)$, length function $\ell_I = \ell\,|_{W_I}$, longest element $w_{\circ,I}$, etc. For example, $W_\varnothing = \{e\}$ while $W_S = W$.

We denote by $W^I := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}$ the set of elements of W with no right descents in I. For example, $W^\varnothing = W$ while $W^S = \{e\}$. Observe that for any $x \in W^I$, we have $x(\Delta_I) \subseteq \Phi^+$ and thus $x(\Phi_I^+) \subseteq \Phi^+$. We will use this property repeatedly in this paper.

Any element $w \in W$ admits a unique factorization $w = w^I \cdot w_I$ with $w^I \in W^I$ and $w_I \in W_I$, and moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$ (see *e.g.,* [BB05, Prop. 2.4.4]). Therefore, W^I is the set of *minimal length coset representatives* of the cosets W/W_I. Throughout the paper, we will always implicitly assume that $x \in W^I$ when writing that xW_I is a *standard parabolic coset*. Note that any standard parabolic coset $xW_I = [x, xw_{\circ,I}]$ is an interval in the weak order. The *Coxeter complex* $\mathcal{P}_W$ is the abstract simplicial complex whose faces are all standard parabolic cosets of W:

$$\mathcal{P}_W = \bigcup_{I \subseteq S} W/W_I = \big\{xW_I \mid I \subseteq S,\ x \in W\big\} = \big\{xW_I \mid I \subseteq S,\ x \in W^I\big\}.$$

We will also need *Deodhar's Lemma*: for $s \in S$, $I \subseteq S$ and $x \in W^I$, either $sx \in W^I$ or $sx = xr$ for some $r \in I$. See *e.g.,* [GP00, Lemma 2.1.2] where it is stated for the cosets $W_I \backslash W$ instead of W/W_I.

1.5 Permutahedron

Remember that a *polytope* P is the convex hull of finitely many points of V, or equivalently a bounded intersection of finitely many affine halfspaces of V. The *faces* of P are the intersections of P with its supporting hyperplanes and the *face lattice* of P is the lattice of its faces ordered by inclusion. The *inner primal cone* of a face F of P is the cone generated by $\{u - v \mid u \in P, v \in F\}$. The *outer normal cone* of a face F of P is the cone generated by the outer normal vectors of the facets of P containing F. Note that these two cones are polar to each other. The *normal fan* is the complete polyhedral fan formed by the outer normal cones of all faces of P. We refer to [Zie95] for details on polytopes, cones, and fans.

The *W-permutahedron* $\mathsf{Perm}^p(W)$ is the convex hull of the orbit under W of a generic point $p \in V$ (not located on any reflection hyperplane of W). Its vertex and facet descriptions are given by

$$\mathsf{Perm}^p(W) = \mathrm{conv}\big\{w(p) \mid w \in W\big\} = \bigcap_{\substack{s \in S \\ w \in W}} \big\{v \in V \mid \langle\, w(\omega_s) \mid v \,\rangle \le \langle\, \omega_s \mid p \,\rangle\big\}.$$

We often write $\mathsf{Perm}(W)$ instead of $\mathsf{Perm}^p(W)$ as the combinatorics of the W-permutahedron is independent of the choice of p and is encoded by the Coxeter complex $\mathcal{P}_W$. More precisely, each standard parabolic coset xW_I corresponds to a face $\mathbf{F}(xW_I)$ of $\mathsf{Perm}^p(W)$ given by

$$\mathbf{F}(xW_I) = x\big(\mathsf{Perm}^p(W_I)\big) = \mathsf{Perm}^{x(p)}\big(xW_Ix^{-1}\big).$$

Therefore, the k-dimensional faces of $\mathsf{Perm}^p(W)$ correspond to the cosets xW_I with $|I| = k$ and the face lattice of $\mathsf{Perm}^p(W)$ is isomorphic to the inclusion poset $(\mathcal{P}_W, \subseteq)$. The normal fan of $\mathsf{Perm}^p(W)$ is the Coxeter fan. The graph of the permutahedron $\mathsf{Perm}^p(W)$ is isomorphic to the Cayley graph of the Coxeter system (W, S). Moreover, when oriented in the linear direction $w_\circ(p) - p$, it coincides with the Hasse diagram of the (right) weak order on W. We refer the reader to [Hoh12] for more details on the W-permutahedron.

2 Facial weak order on the Coxeter complex

In this section we study an analogue of the weak order on standard parabolic cosets, which we call the *facial weak order*. It was defined in type A by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer in [KLN$^+$01], then extended for arbitrary finite types by P. Palacios and M. Ronco in [PR06].

Definition 1 ([PR06]) *The (right) facial weak order is the order $\le$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types: for $I \subseteq S$ and $x \in W^I$,*

$$\text{(1)} \qquad xW_I \lessdot xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$
$$\text{(2)} \qquad xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I.$$

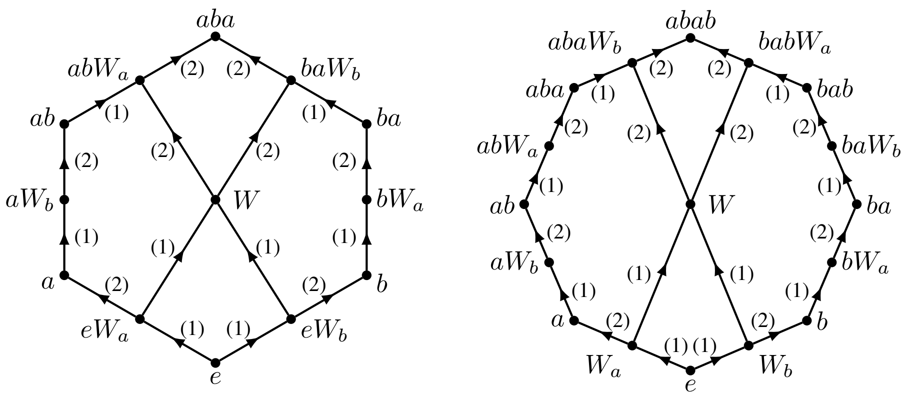


Fig. 1: The facial weak order on the standard parabolic cosets of the Coxeter group of types A_2 and B_2. Edges are labelled with the cover relations of type (1) or (2) as in Definition 1.

Remark 2 *1. These cover relations translate to the following geometric conditions on faces of the permutahedron* $\mathsf{Perm}(W)$*: a face F is covered by a face G if and only if either F is a facet of G with the same weak order minimum, or G is a facet of F with the same weak order maximum.*

2. *Under the inclusion $x \mapsto xW_\varnothing$ from W to $\mathcal{P}_W$, we will see in Corollary 11 that the restriction of the (right) facial weak order to the vertices of $\mathcal{P}_W$ is the (right) weak order. It is not obvious at first sight from Definition 1.*
3. *It is known that for $I \subseteq S$ the set of minimal length coset representatives W^I has a maximal length element $w_\circ w_{\circ,I}$. The element $w_{\circ,I}w_{\circ,I\smallsetminus\{s\}}$ is therefore the maximal length element of the set $W_I^{I\smallsetminus\{s\}} = W_I \cap W^{I\smallsetminus\{s\}}$, which is the set of minimal coset representatives of the cosets $W_I/W_{I\smallsetminus\{s\}}$, see [GP00, Section 2.2] for more details.*

This paper gives two convenient equivalent definitions for the facial weak order (see Section 2.2). The first uses sets of roots (see Section 2.1) to generalize the geometric characterization of the weak order with inversion sets. The second one uses weak order comparisons on the minimal and maximal representatives of the cosets. The advantage of these definitions is that they give immediate global comparison, whereas the original definition of [PR06] uses cover relations. We use these new characterizations of the facial weak order to prove that this poset is in fact a lattice (see Section 2.3).

2.1 Root and weight inversion sets of standard parabolic cosets

We now define a collection of roots and a collection of weights associated to each standard parabolic coset. The notion of root inversion sets of standard parabolic cosets generalizes the inversion sets of elements of W (see Proposition 6). We will use root inversion sets extensively for our study of the facial weak order. In contrast, weight inversion sets are not as essential and could be ignored for a first reading. We define them here as they are polar to the root inversion sets and appear naturally in our geometric intuition of the W-Coxeter arrangement and of the W-permutahedron (see Proposition 4).

Definition 3 *The root inversion set $\mathbf{R}(xW_I)$ and weight inversion set $\mathbf{W}(xW_I)$ of a standard parabolic coset xW_I are respectively defined by*

$$\mathbf{R}(xW_I) := x\big(\Phi^- \cup \Phi_I^+\big) \;\subseteq\; \Phi \qquad \textit{and} \qquad \mathbf{W}(xW_I) := x\big(\nabla_{S \smallsetminus I}\big) \;\subseteq\; \Omega.$$

The following statement gives the precise connection to the geometry of the W-permutahedron and is illustrated on Figure 2 for the Coxeter group of type A_2.

Proposition 4 *Let xW_I be a standard parabolic coset of W. Then*
(i) $\mathrm{cone}(\mathbf{R}(xW_I))$ *is the inner primal cone of the face* $\mathbf{F}(xW_I)$ *of* $\mathsf{Perm}(W)$*,*
(ii) $\mathrm{cone}(\mathbf{W}(xW_I))$ *is the outer normal cone of the face* $\mathbf{F}(xW_I)$ *of* $\mathsf{Perm}(W)$*,*
(iii) the cones generated by the root inversion set and by the weight inversion set of xW_I are polar to each other: $\mathrm{cone}(\mathbf{R}(xW_I))^\diamond = \mathrm{cone}(\mathbf{W}(xW_I))$.

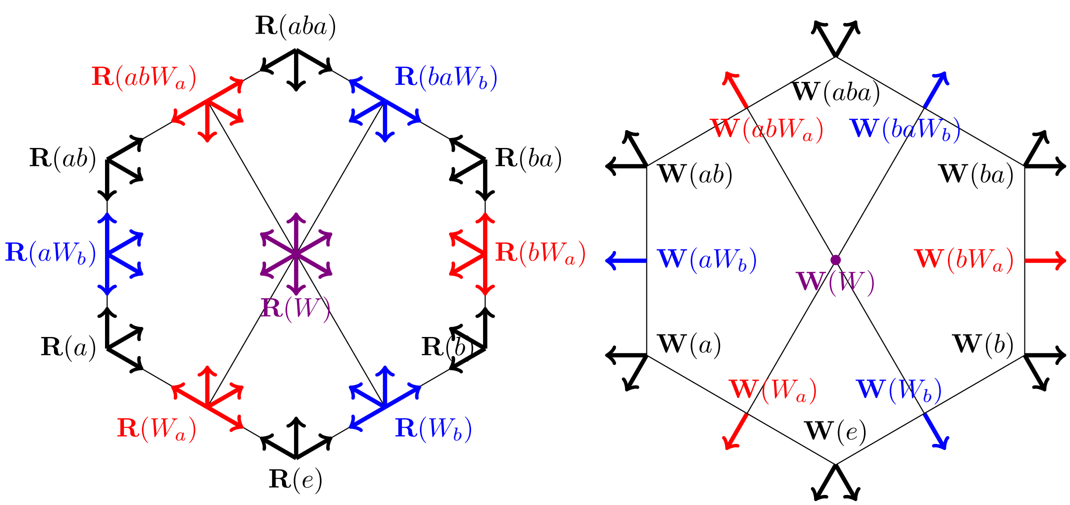


Fig. 2: The root inversion sets (left) and weight inversion sets (right) of the A_2 standard parabolic cosets. Note that positive roots point downwards.

It is well-known that the map $\mathbf{N}$, sending an element $w \in W$ to its inversion set $\mathbf{N}(w) = \Phi^+ \cap w(\Phi^-)$ is injective, see *e.g.,* [HL16, Section 2]. The following corollary is the analog for the maps $\mathbf{R}$ and $\mathbf{W}$.

Corollary 5 *The maps $\mathbf{R}$ and $\mathbf{W}$ are both injective.*

Our next statement connects the root inversion set $\mathbf{R}(xW_\varnothing)$ to the inversion set and reduced words of $x \in W$. For brevity we write $\mathbf{R}(x)$ instead of $\mathbf{R}(xW_\varnothing)$.

Proposition 6 *For any $x \in W$, we have:*
(i) $\mathbf{R}(x) = \mathbf{N}(x) \cup -\big(\Phi^+ \smallsetminus \mathbf{N}(x)\big)$ *where* $\mathbf{N}(x) = \Phi^+ \cap x(\Phi^-)$ *is the (left) inversion set of* x*.*
(ii) If $x = s_1 s_2 \cdots s_k$ *is reduced, then* $\mathbf{R}(xW_\varnothing) = \Phi^- \triangle \{\pm\alpha_{s_1}, \pm s_1(\alpha_{s_2}), \dots, \pm s_1 \cdots s_{k-1}(\alpha_{s_k})\}$*.*
(iii) $\mathbf{R}(xw_\circ) = -\mathbf{R}(x)$ *and* $\mathbf{R}(w_\circ x) = w_\circ\big(\mathbf{R}(x)\big)$*.*

The next statement gives a characterization of the weak order on W in terms of root inversion sets, which generalizes the characterization of the weak order in term of inversion sets.

Corollary 7 *For $x, y \in W$, we have*

$$\begin{aligned} x \le y \iff & \mathbf{R}(x) \smallsetminus \mathbf{R}(y) \subseteq \Phi^- \quad \textit{and} \quad \mathbf{R}(y) \smallsetminus \mathbf{R}(x) \subseteq \Phi^+, \\ \iff & \mathbf{R}(x) \cap \Phi^+ \subseteq \mathbf{R}(y) \cap \Phi^+ \quad \textit{and} \quad \mathbf{R}(x) \cap \Phi^- \supseteq \mathbf{R}(y) \cap \Phi^-. \end{aligned}$$

Finally, we observe that the root and weight inversion sets of a parabolic coset xW_I can be computed from that of its minimal and maximal length representatives x and $xw_{\circ,I}$.

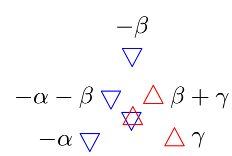
Proposition 8 *The root and weight inversion sets of xW_I can be computed from those of x and $xw_{\circ,I}$ by*

$$\mathbf{R}(xW_I) = \mathbf{R}(x) \cup \mathbf{R}(xw_{\circ,I}) \qquad \textit{and} \qquad \mathbf{W}(xW_I) = \mathbf{W}(x) \cap \mathbf{W}(xw_{\circ,I}).$$

2.2 Two alternative characterizations of the facial weak order

Using the root inversion sets defined in the previous section, we now give two equivalent characterizations of the facial weak order defined by P. Palacios and M. Ronco in [PR06] (see Definition 1). In type A, the equivalence (i) $\iff$ (ii) below is stated in [KLN$^+$01, Theorem 5] in terms of half-inversion tables. We have illustrated the facial weak order by the means of root inversion sets in Figure 3.

Remark 9 *In Figure 3, each face is labelled by its root inversion set. To visualize the roots, we consider the affine plane P passing through the simple roots $\{\alpha, \beta, \gamma\}$. A positive (resp. negative) root ρ is then seen as a red upward (resp. blue downward) triangle placed at the intersection of $\mathbb{R}\rho$ with the plane P. For instance, the root set* $\mathbf{R}(cbW_a) = \{\gamma, \beta+\gamma, \alpha+\beta+\gamma\} \cup \{-\alpha, -\beta, -\alpha-\beta-\gamma, -\alpha-\beta\}$ *is represented on the right. The star in the middle represents both $\alpha+\beta+\gamma$ and $-\alpha-\beta-\gamma$.*



Theorem 10 *The following conditions are equivalent for two standard parabolic cosets $xW_I, yW_J \in \mathcal{P}_W$:*
(i) $xW_I \le yW_J$ *in facial weak order,*
(ii) $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$*,*
(iii) $x \le y$ *and* $xw_{\circ,I} \le yw_{\circ,J}$ *in weak order.*

Proof of Theorem 10 : We give a brief outline of the proof. The implication (i)⇒(iii) is immediate by looking at each cover relation separately.

Combining Corollary 7 and Proposition 8 gives the implication (iii)⇒(ii).

Finally, for the implication (ii)⇒(i) we consider two standard parabolic cosets xW_I and yW_J which satisfy (ii) and construct a path of cover relations as in Definition 1 between them. We proceed by induction on the cardinality $|\,\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)|$. If $|\,\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)| = 0$, then $\mathbf{R}(xW_I) = \mathbf{R}(yW_J)$,

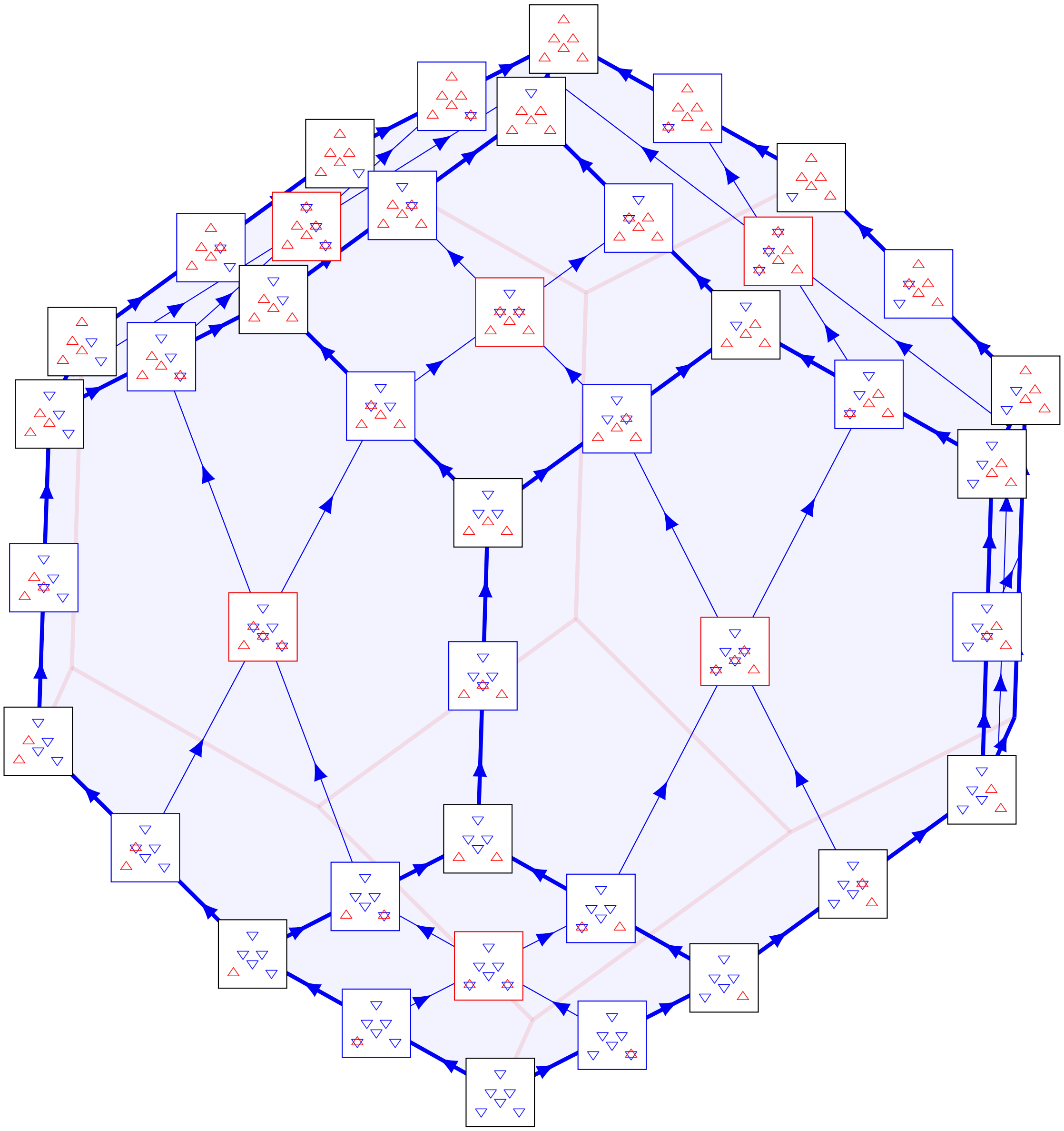


Fig. 3: The facial weak order on the standard parabolic cosets of the Coxeter group of type A_3. Each coset xW_I is replaced by its root inversion set $\mathbf{R}(xW_I)$, represented as follows: down blue triangles stand for negative roots while up red triangles stand for positive roots, and the position of each triangle is given by the barycentric coordinates of the corresponding root with respect to the three simple roots (α_1 on bottom left, α_2 on top, and α_3 on bottom right);see Remark 9 for a more detailed discussion.

which ensures that $xW_I = yW_J$ by Corollary 5. Assume now that $|\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)| > 0$. So either $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \neq \varnothing$ or $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \neq \varnothing$. We consider only the first case, the other one being symmetric. To proceed by induction, we find a new coset zW_K so that:

- $xW_I \lessdot zW_K$ is one of the cover relations of Definition 1,
- zW_K and yW_J still satisfy (ii), and
- $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J) \subsetneq \mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$.

The new coset is constructed by adding or deleting at least one root from $\mathbf{R}(xW_I)$. We first show that there exists $s \in S$ such that $-x(\alpha_s) \notin \mathbf{R}(yW_J)$. We then fix such an s and look at two cases, on whether or not $s \in I$. Each case produces one of the two cover relations in P. Palacios and M. Ronco's Definition 1. As each case produces a cover relation we see that by induction we have a path from xW_I to yW_J as we already have a path from zW_K to yW_J and a cover between xW_I and zW_K. □

Using our characterization (ii) of Theorem 10 together with Corollary 7, we obtain that the (right) facial weak order and the (right) weak order coincide on the elements of W. Note that this is not obvious with the cover relations from Definition 1.

Corollary 11 *For any $x, y \in W$, we have $x \le y$ in (right) weak order if and only if $xW_\varnothing \le yW_\varnothing$ in (right) facial weak order.*

The weak order anti-automorphisms $x \mapsto xw_\circ$ and $x \mapsto w_\circ x$ and the automorphism $x \mapsto w_\circ x w_\circ$ carry out on standard parabolic cosets. The following statement gives the precise definitions of these maps.

Proposition 12 *The maps $xW_I \longmapsto w_\circ x w_{\circ,I} W_I$ and $xW_I \longmapsto x w_{\circ,I} w_\circ W_{w_\circ I w_\circ}$ are anti-automorphisms of the weak order on parabolic cosets of W. Consequently, the map $xW_I \longmapsto w_\circ x w_\circ W_{w_\circ I w_\circ}$ is an automorphism of the weak order on parabolic cosets of W.*

2.3 The facial weak order is a lattice

In this section, we show that the facial weak order on standard parabolic cosets is actually a lattice. It generalizes the result for the symmetric group due to D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer [KLN+01] to the facial weak order on arbitrary finite Coxeter groups introduced by P. Palacios and M. Ronco [PR06]. The characterizations of the facial weak order given in Theorem 10 are key here.

Theorem 13 *The facial weak order $(\mathcal{P}_W, \le)$ is a lattice. The meet and join of two standard parabolic cosets xW_I and yW_J are given by:*

- $xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$, *where* $z_\wedge = x \wedge y$ *and* $K_\wedge = D_L\big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\big)$;
- $xW_I \vee yW_J = z_\vee W_{K_\vee}$, *where* $z_\vee = xw_{\circ,I} \vee yw_{\circ,J}$ *and* $K_\vee = D_L\big(z_\vee^{-1}(x \vee y)\big)$.

Note that in the second point of the previous statement, the minimal representative of the coset $z_\vee W_{K_\vee}$ is in fact $z_\vee w_{\circ,K_\vee}$, not $z_\vee$. Unlike in the rest of the paper, we take the liberty to use another coset representative than the minimal one to underline the symmetry between meet and join in facial weak order.

Example 14 *Before proving the above statement, we give an example of a computation of the meet in the facial weak order. Consider the Coxeter system $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^2 = 1 \rangle$ of type A_3. To find the meet of $cbaW_{bc}$ and $acbW_\varnothing$, we compute:*

$$z_\wedge = cba \wedge acb = c,$$
$$K_\wedge = D_L\big(z_\wedge^{-1}(cbaw_{\circ,bc} \wedge acbw_{\circ,\varnothing})\big) = D_L\big(c(cbabcb \wedge acb)\big) = D_L(c(acb)) = \{a\}.$$

Thus we have that $cbaW_{bc} \wedge acbW_\varnothing = z_\wedge W_{K_\wedge} = cW_a$.

Proof of Theorem 13: Throughout the proof we use the characterization of the facial weak order given in Theorem 10 (iii): $xW_I \le yW_J \iff x \le y$ and $xw_{\circ,I} \le yw_{\circ,J}$. We first prove the existence of the meet, then use Proposition 12 to deduce the existence and formula for the join.

Existence of meet. For any $s \in K_\wedge$, we have

$$\ell(xw_{\circ,I} \wedge yw_{\circ,J}) - \ell(sz_\wedge^{-1}) \le \ell\left(sz_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\right) = \ell(xw_{\circ,I} \wedge yw_{\circ,J}) - \ell(z_\wedge^{-1}) - 1.$$

Therefore, we have $z_\wedge \in W^{K_\wedge}$.

Since $K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$, we have $w_{\circ,K_\wedge} \le z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})$. Since $z_\wedge \in W^{K_\wedge}$, we obtain $z_\wedge w_{\circ,K_\wedge} \le xw_{\circ,I} \wedge yw_{\circ,J}$. We thus have $z_\wedge = x \wedge y \le x$ and $z_\wedge w_{\circ,K_\wedge} \le xw_{\circ,I} \wedge yw_{\circ,J} \le xw_{\circ,I}$, which implies $z_\wedge W_{K_\wedge} \le xW_I$, by Theorem 10 (ii). By symmetry, $z_\wedge W_{K_\wedge} \le yW_J$.

It remains to show that $z_\wedge W_{K_\wedge}$ is the greatest lower bound. Consider a standard parabolic coset zW_K such that $zW_K \le xW_I$ and $zW_K \le yW_J$. We want to show that $zW_K \le z_\wedge W_{K_\wedge}$, that is, $z \le z_\wedge$ and $zw_{\circ,K} \le z_\wedge w_{\circ,K_\wedge}$. The first inequality is immediate since $z \le x$ and $z \le y$ so that $z \le x \wedge y = z_\wedge$. For the second one, we consider the reduced words $x = zx'$, $y = zy'$, and $z_\wedge = zz'_\wedge$ where $z'_\wedge = x' \wedge y'$. Since $zw_{\circ,K} \le xw_{\circ,I}$ and $zw_{\circ,K} \le yw_{\circ,J}$, we have

$$zw_{\circ,K} \le xw_{\circ,I} \wedge yw_{\circ,J} = zx'w_{\circ,I} \wedge zy'w_{\circ,J} = z(x'w_{\circ,I} \wedge y'w_{\circ,J}).$$

Thus $w_{\circ,K} \le x'w_{\circ,I} \wedge y'w_{\circ,J}$, since all words are reduced here. Therefore $K \subseteq D_L(x'w_{\circ,I} \wedge y'w_{\circ,J})$.

We now claim that $D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}) \subseteq D_L(z'_\wedge w_{\circ,K_\wedge})$. To see it, consider $s \in D_L(x'w_{\circ,I} \wedge y'w_{\circ,J})$ and assume by contradiction that $s \notin D_L(z'_\wedge w_{\circ,K_\wedge})$. Then s does not belong to $D_L(z'_\wedge)$, since the expression $z'_\wedge w_{\circ,K_\wedge}$ is reduced. By Deodhar's Lemma (see Section 1.4) we obtain that either $sz'_\wedge \in W^{K_\wedge}$ or $sz'_\wedge = z'_\wedge t$ where

$$t \in D_L\big(z'^{-1}_\wedge(x'w_{\circ,I} \wedge y'w_{\circ,J})\big) = D_L\big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\big) = K_\wedge.$$

In the first case we obtain

$$1 + \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) = \ell(sz'_\wedge w_{\circ,K_\wedge}) = \ell(z'_\wedge w_{\circ,K_\wedge}) - 1 = \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) - 1$$

a contradiction. In the second case, we get

$$1 + \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) = \ell(sz'_\wedge w_{\circ,K_\wedge}) = \ell(z'_\wedge) + \ell(tw_{\circ,K_\wedge}) = \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) - 1,$$

a contradiction again. This proves that $D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}) \subseteq D_L(z'_\wedge w_{\circ,K_\wedge})$.

To conclude the proof, we get from $K \subseteq D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}) \subseteq D_L(z'_\wedge w_{\circ,K_\wedge})$ that $w_{\circ,K} \le z'_\wedge w_{\circ,K_\wedge}$, and finally that $zw_{\circ,K} \le zz'_\wedge w_{\circ,K_\wedge} = z_\wedge w_{\circ,K_\wedge}$ since all expressions are reduced. Since $z \le z_\wedge$ and $zw_{\circ,K} \le z_\wedge w_{\circ,K_\wedge}$, we have $zW_K \le z_\wedge W_{K_\wedge}$ so that $z_\wedge W_{K_\wedge}$ is indeed the greatest lower bound.

Existence of join. The existence and the formula for the join follow from that of the meet, using the anti-automorphism $\Psi : xW_I \longmapsto w_\circ xw_{\circ,I}W_I$ from Proposition 12. □

Remark 15 *It is well-known that the map $x \mapsto xw_\circ$ is an orthocomplementation of the weak order: it is involutive, order-reversing and satisfies $xw_\circ \wedge x = e$ and $xw_\circ \vee x = w_\circ$. In other words, it endows the weak order with a structure of ortholattice; see for instance [BB05, Corollary 3.2.2]. This is not anymore the case for the facial weak order: the map $xW_I \longmapsto w_\circ xw_{\circ,I}W_I$ is indeed involutive and order-reversing, but is not an orthocomplementation: for a (counter-)example, consider $x = e$ and $I = S$.*

3 Lattice congruences of the facial weak order

In this section we observe that any lattice congruence $\equiv$ of the weak order extends to a lattice congruence $\equiv_{\mathcal{P}}$ of the facial weak order. Let us first recall the definition of lattice congruences, see *e.g.*, [Rea05].

Definition 16 *An order congruence is an equivalence relation $\equiv$ on a poset P such that:*
(i) Every equivalence class under $\equiv$ is an interval of P.
(ii) The projection $\pi^{\uparrow} : P \to P$ (resp. $\pi_{\downarrow} : P \to P$), which maps an element of P to the maximal (resp. minimal) element of its equivalence class, is order preserving.
The quotient $P/\equiv$ is a poset on the equivalence classes of $\equiv$, where the order relation is defined by $X \le Y$ in $P/\equiv$ if and only if there exists representatives $x \in X$ and $y \in Y$ such that $x \le y$ in P. If P is moreover a finite lattice, then $\equiv$ is automatically a lattice congruence, i.e., for any $x \equiv x'$ and $y \equiv y'$, we have $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. The poset quotient $P/\equiv$ then inherits a lattice structure.

Consider a lattice congruence $\equiv$ of the weak order on W whose up and down projections are denoted by $\pi^{\uparrow}$ and $\pi_{\downarrow}$ respectively. We want to extend $\equiv$ to a lattice congruence $\equiv_{\mathcal{P}}$ of the facial weak order on $\mathcal{P}_W$. We need the following technical statement.

Lemma 17 *For any coset xW_I, there are unique subsets $\Sigma^{\uparrow}(x,I)$ of $S \smallsetminus D_R\big(\pi^{\uparrow}(x)\big)$ and $\Sigma_{\downarrow}(x,I)$ of $D_R\big(\pi_{\downarrow}(xw_{\circ,I})\big)$ such that $xw_{\circ,I} \le \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)} \le \pi^{\uparrow}(xw_{\circ,I})$ and $\pi_{\downarrow}(x) \le \pi_{\downarrow}(xw_{\circ,I})w_{\circ,\Sigma_{\downarrow}(x,I)} \le x$.*

Based on this lemma, we define two maps $\Pi^{\uparrow} : \mathcal{P}_W \to \mathcal{P}_W$ and $\Pi_{\downarrow} : \mathcal{P}_W \to \mathcal{P}_W$ by

$$\Pi^{\uparrow}(xW_I) = \pi^{\uparrow}(x)W_{\Sigma^{\uparrow}(x,I)} \qquad \text{and} \qquad \Pi_{\downarrow}(xW_I) = \pi_{\downarrow}(xw_{\circ,I})W_{\Sigma_{\downarrow}(x,I)}.$$

We again write $\Pi_{\downarrow}(xW_I) = \pi_{\downarrow}(xw_{\circ,I})W_{\Sigma_{\downarrow}(x,I)}$ instead of $\Pi_{\downarrow}(xW_I) = \pi_{\downarrow}(xw_{\circ,I})\, w_{\circ,\Sigma^{\uparrow}(x,I)}W_{\Sigma_{\downarrow}(x,I)}$ to make apparent the symmetry between $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$.

Theorem 18 *The maps $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$ fulfill the following properties:*
(i) $\Pi_{\downarrow}(xW_I) \le xW_I \le \Pi^{\uparrow}(xW_I)$ for any coset xW_I.
(ii) $\Pi^{\uparrow} \circ \Pi^{\uparrow} = \Pi^{\uparrow} \circ \Pi_{\downarrow} = \Pi^{\uparrow}$ and $\Pi_{\downarrow} \circ \Pi_{\downarrow} = \Pi_{\downarrow} \circ \Pi^{\uparrow} = \Pi_{\downarrow}$.
(iii) $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$ are order preserving.
Therefore, the fibers of $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$ coincide and define a lattice congruence $\equiv_{\mathcal{P}}$ of the facial weak order.

N. Reading proved in [Rea05] that $\equiv$ naturally defines a complete simplicial fan which coarsens the Coxeter fan. Namely, for each congruence class γ of $\equiv$, consider the cone C_γ obtained by glueing the maximal chambers $\operatorname{cone}(x(\nabla))$ of the Coxeter fan corresponding to the elements x in γ. It turns out that each of these cones C_γ is convex and that the collection of cones $\{C_\gamma \mid \gamma \in W/\equiv\}$, together with all their faces, form a complete simplicial fan which we denote by $\mathcal{F}_{\equiv}$. We now use the congruence $\equiv_{\mathcal{P}}$ of the facial weak order to describe all cones of $\mathcal{F}_{\equiv}$ (not only the maximal ones). This shows that the lattice structure on the maximal faces of $\mathcal{F}_{\equiv}$ extends to a lattice structure on all cones of the fan $\mathcal{F}_{\equiv}$. As in Section 2.1, we first introduce the root and weight inversion sets of the congruence classes of $\equiv_{\mathcal{P}}$.

Definition 19 *The root inversion set $\mathbf{R}(\Gamma)$ and the weight inversion set $\mathbf{W}(\Gamma)$ of a congruence class Γ of $\equiv_{\mathcal{P}}$ are defined by $\mathbf{R}(\Gamma) = \bigcap_{xW_I \in \Gamma} \mathbf{R}(xW_I)$ and $\mathbf{W}(\Gamma) = \bigcup_{xW_I \in \Gamma} \mathbf{W}(xW_I)$.*

Proposition 20 *For any two congruence classes Γ, Γ' of $\equiv_{\mathcal{P}}$, we have $\Gamma \le \Gamma'$ in the quotient of the facial weak order by $\equiv_{\mathcal{P}}$ if and only if $\mathbf{R}(\Gamma) \smallsetminus \mathbf{R}(\Gamma') \subseteq \Phi^-$ and $\mathbf{R}(\Gamma') \smallsetminus \mathbf{R}(\Gamma) \subseteq \Phi^+$.*

Theorem 21 *The collection of cones $\{\operatorname{cone}(\mathbf{W}(\Gamma)) \mid \Gamma \in \mathcal{P}_W/{\equiv_{\mathcal{P}}}\}$ forms the complete simplicial fan $\mathcal{F}_{\equiv}$.*

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