A combinatorial approach to Macdonald q, t-symmetry via the Carlitz bijection

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Abstract. We investigate the combinatorics of the symmetry relation $\widetilde{H}_\mu(\mathbf{x}; q, t) = \widetilde{H}_{\mu^*}(\mathbf{x}; t, q)$ on the transformed Macdonald polynomials, from the point of view of the combinatorial formula of Haglund, Haiman, and Loehr in terms of the inv and maj statistics on Young diagram fillings. By generalizing the Carlitz bijection on permutations, we provide a purely combinatorial proof of the relation in the case of Hall-Littlewood polynomials ($q = 0$) for the coefficients of the square-free monomials in the variables $\mathbf{x}$. Our work in this case relates the Macdonald inv and maj statistics to the monomial basis of the modules R_μ studied by Garsia and Procesi. We also provide a new proof for the full Macdonald relation in the case when μ is a hook shape.

Résumé. Nous investiguons la combinatoire de la relation $\widetilde{H}_\mu(\mathbf{x}; q, t) = \widetilde{H}_{\mu^*}(\mathbf{x}; t, q)$ des polynômes (transformés) de Macdonald, du point de vue de la formule de Haglund, Haiman et Loehr, exprimant ces polynômes en termes d'inversions et d'indices majeurs de remplissages de diagrammes de Young. Nous généralisons la bijection de Carlitz sur les permutations, ce qui nous permet de déduire cette relation pour les polynômes de Hall-Littlewood (le cas $q = 0$), de façon purement combinatoire, pour les monômes sans carrés dans les variables $\mathbf{x}$. Notre approche lie les statistiques de Macdonald inv et maj à la base monomiale des modules R_μ de Garsia-Procesi. Nous fournissons aussi une nouvelle démonstration de la relation (complète) de Macdonald lorsque μ est une équerre.

Keywords. Macdonald polynomials, Hall-Littlewood polynomials, symmetric functions, Young tableaux, cocharge, Mahonian statistics

1 Introduction

Let $\Lambda_{q,t}(x)$ denote the ring of symmetric polynomials in the countably many indeterminates $x_1, x_2, \ldots$, with coefficients in the field $\mathbb{Q}(q, t)$ of rational functions in two variables. The *(transformed) Macdonald polynomials* $\widetilde{H}_\mu(x; q, t) \in \Lambda_{q,t}(x)$, indexed by the set of all partitions μ, form an orthogonal basis of $\Lambda_{q,t}(x)$, and have specializations $\widetilde{H}_\mu(x; 0, 1) = h_\mu$ and $\widetilde{H}_\mu(x; 1, 1) = e_1^{|\mu|}$, where h_λ and e_λ are the homogeneous and elementary symmetric functions, respectively. The polynomials $\widetilde{H}_\mu$ are a transformation of the functions P_λ originally defined by Macdonald in [13], and have been the subject of much recent attention in combinatorics and algebraic geometry. [6] [8] [9]

Fig. 1: A filling of a Young diagram. Descents are shown in bold, and attacking pairs are connected by a segment.

The symmetric functions $\widetilde{H}_\mu$ may be defined as the unique collection of polynomials that satisfy certain triangularity conditions. However, an explicit combinatorial formula for the transformed Macdonald polynomials $\widetilde{H}_\mu$ was discovered in [8] in 2004. The formula is

$$\widetilde{H}_\mu(x; q, t) = \sum_\sigma q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma, \tag{1}$$

where the sum ranges over all fillings σ of the diagram of μ with positive integers, and x^σ is the monomial $x_1^{m_1} x_2^{m_2} \cdots$ where m_i is the number of times the letter i occurs in σ. The statistics inv and maj are generalizations of the Mahonian statistics inv and maj for permutations. Their precise definitions can be stated as follows.

Definition 1 *Given a word $w = w_1 \cdots w_n$ where the letters w_i are taken from some partially ordered alphabet A, a **descent** of w is an index i for which $w_i > w_{i+1}$. The **major index** of w, denoted $\mathrm{maj}(w)$, is the sum of the descents of w.*

Definition 2 *Given a filling σ of a Young diagram of shape μ drawn in French notation[(i)], let $w^{(1)}, \ldots, w^{(\mu_1)}$ be the words formed by the successive columns of σ, read from top to bottom. Then*

$$\mathrm{maj}(\sigma) = \sum_s \mathrm{maj}(w^{(s)}).$$

Example 1 *The major index of the filling in Figure 1 is 5, since the first column has major index 4, the second has major index 0, and the third column, 1.*

For the statistic inv, we use the notion of the ***arm*** of an entry, defined as the number of squares strictly to the right of the entry. A ***descent*** is an entry which is strictly greater than the entry just below it.

Definition 3 *An **attacking pair** in a filling σ of a Young diagram is a pair of entries u and v with $u > v$ satisfying one of the following conditions:*

1. *u and v are in the same row, with u to the left of v, or*

2. *u is in the row above v and strictly to its right.*

Definition 4 *The quantity $\mathrm{inv}(\sigma)$ is defined as $\mathrm{inv}(\sigma) = \#(\text{attacking pairs}) - \sum(\text{arms of descents})$. For a word w, the **inversion number** $\mathrm{inv}(w)$ is equal to inv of the corresponding one-row filling.*

[(i)] The Young diagram in Figure 1 has partition shape $\mu = (3, 3, 2, 1)$. French notation indicates that the rows in the diagram have lengths equal to the parts of the partition, listed from bottom to top.

Example 2 *In Figure 1, there are* 4 *attacking pairs, and the arms of the descents have lengths* 0*,* 2*, and* 0*. Thus* $\text{inv}(\sigma) = 4 - 2 = 2$.

The well-known q, t-symmetry relation for the Macdonald polynomials $\widetilde{H}_\mu(x; q, t)$ states that

$$\widetilde{H}_\mu(x; q, t) = \widetilde{H}_{\mu^*}(x; t, q).$$

This is a result of the triangularity conditions that define $\widetilde{H}_\mu$, and is also clear from Haiman's geometric interpretation [10]. When combined with the combinatorial formula, we obtain a remarkable identity:

$$\sum_{\sigma: \mu \to \mathbb{Z}_+} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)} x^\sigma = \sum_{\rho: \mu^* \to \mathbb{Z}_+} q^{\text{maj}(\rho)} t^{\text{inv}(\rho)} x^\rho. \tag{2}$$

Setting $t = 1$ and $\mu = (n)$ and taking the coefficient of $x_1 \cdots x_n$ on both sides, this reduces to

$$\sum_{w \in S_n} q^{\text{inv}(w)} = \sum_{w \in S_n} q^{\text{maj}(w)},$$

which demonstrates the well-known equidistribution of the Mahonian statistics inv and maj on permutations. There are several known bijective proofs of this simpler identity (see [1], [3], [14]).

In light of this, it is natural to ask if there is an elementary combinatorial proof of (2), in the sense of Conjecture 1 below. To state the conjecture we first introduce some notation that we will use throughout.

Definition 5 *The* **content** *of a filling σ, denoted $|\sigma|$, is the sequence $\alpha = (\alpha_1, \cdots, \alpha_k)$ where α_i is the number of i's used in the filling. We also define the symbols:*

- $\mathcal{F}$ *- set of all fillings of Young diagrams with positive integers*

- $\mathcal{F}_\mu^\alpha$ *- set of fillings of shape μ and content α*

- $\mathcal{F}_\mu^\alpha|_{\text{inv}=a}$, $\mathcal{F}_\mu^\alpha|_{\text{maj}=b}$ *- set of fillings $\sigma \in \mathcal{F}_\mu^\alpha$ for which $\text{inv}(\sigma) = a$ or $\text{maj}(\sigma) = b$ respectively.*

Definition 6 *We define a* **weighted set** *to be a set S equipped with a list of statistics $\text{stat}_i : S \to \mathbb{Z}$, and a* **morphism of weighted sets** *to be a map that preserves their statistics. We write $(S; \text{stat}_1, \text{stat}_2, \ldots)$ to denote the weighted set if the statistics are not understood.*

Conjecture 1 *There is an explicit isomorphism of weighted sets*

$$\varphi : (\mathcal{F}; \text{inv}, \text{maj}) \to (\mathcal{F}; \text{maj}, \text{inv})$$

which interchanges inv *and* maj *and sends a partition shape to its conjugate.*

In [5], the author provides explicit bijections φ for several infinite families of values of a, b, α, and μ. Our bijections naturally extend Carlitz's bijection on permutations, which is defined in section 2. In this abridged version, we give bijections in two of the cases explored in [5], and refer to [5] for all proofs.

In Section 3, we give an explicit bijection φ in the case that μ is a hook shape. In Section 4, we investigate the Hall-Littlewood specialization $a = 0$, which corresponds to setting $q = 0$ in the Macdonald polynomials. We give a bijection in this case for fillings having content $\alpha = (1, 1, \ldots, 1)$, and connect our results to the work of Garsia and Procesi [4], and of Killpatrick [11] on the *cocharge* statistic.

2 The Carlitz bijection

Our approach to the symmetry problem is motivated by Carlitz's bijection $(S_n; \mathrm{inv}) \to (S_n; \mathrm{maj})$, an alternative to the better-known Foata bijection. A full proof of this bijection can be found in Carlitz's original paper [1], or in a somewhat cleaner form in [14]. We briefly recall the definition here.

Definition 7 *A **Carlitz code** of length n is a word $w = w_1 \cdots w_n$ consisting of nonnegative integers such that $w_{n-i} \le i$ for all i. Let C_n denote the set of all Carlitz codes of length n, equipped with the combinatorial statistic Σ taking a word to the sum of its entries.*

Notice that there are $n!$ Carlitz codes of length n. The ***Carlitz bijection*** is the composite

$$(S_n; \mathrm{inv}) \xrightarrow{\;\mathrm{invcode}\;} (C_n; \Sigma) \xrightarrow{\;\mathrm{majcode}^{-1}\;} (S_n; \mathrm{maj})$$

of two isomorphisms of weighted sets, defined as follows.

Definition 8 *The **inversion code** of a permutation π is the sequence $\mathrm{invcode}(\pi) = c_1, \ldots, c_n$ where c_i is the number of attacking pairs, or **inversions**, of the form (j, i) with $i < j$ and $\pi^{-1}(j) < \pi^{-1}(i)$.*

Definition 9 *The **major index code** of a permutation π, denoted $\mathrm{majcode}(\pi)$, is the sequence $c_1, \ldots, c_n$ defined as follows. Given $\pi \in S_n$ written in list notation, successively remove the entries $n, n-1, n-2, \ldots$ and let c_i be the amount the major index decreases at the ith step.*

Example 3 *We have $\mathrm{invcode}(4132) = 1210$. Indeed, the 1 is the smaller entry of one inversion $(4, 1)$, the 2 is the smaller entry of the two inversions $(3, 2)$ and $(4, 2)$, the 3 is the smaller entry of the inversion $(4, 3)$, and the 4 is not the smaller entry of any inversion.*

Let $\pi = 3241$. We have $\mathrm{maj}(\pi) = 1 + 3 = 4$. Removing the 4 leaves the permutation 321, which has major index 3, so $c_1 = 4 - 3 = 1$. Removing the 3 results in 21, which has major index 1, so $c_2 = 3 - 1 = 2$. Continuing in this fashion we find $\mathrm{majcode}(\pi) = 1210$.

Therefore $\mathrm{majcode}^{-1} \circ \mathrm{invcode}(4132) = \mathrm{majcode}^{-1}(1210) = 3241$.

2.1 Carlitz bijection on words

We now generalize the Carlitz bijection to words, i.e. to one-column shapes μ with *any* content α.

Definition 10 *Let $A = (a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots, a_k^{\alpha_k})$ be any finite multiset of size n, with an ordering "$<$" such that $a_1 < a_2 < \cdots < a_k$. We say that a word c of length n is A-**weakly increasing** if every subword of the form*

$$c_{\alpha_1 + \cdots + \alpha_i}, c_{\alpha_1 + \cdots \alpha_i + 1}, c_{\alpha_1 + \cdots \alpha_i + 2}, \ldots, c_{\alpha_1 + \cdots + \alpha_i + \alpha_{i+1} - 1}$$

is weakly increasing.

For instance, if $A = \{1, 1, 2, 3, 3, 3, 4, 4\}$, ordered by magnitude, then the word 23711213 is A-weakly increasing, since the subwords 23, 7, 112, and 13, corresponding to each letter of A, are weakly increasing.

Definition 11 *Let $C_{(1^n), A}$ denote the subset of C_n consisting of all Carlitz codes of length n which are A-weakly increasing. This subset inherits the Σ statistic from C_n.*

We also will make use of Macdonald symmetry in the variables x_i by defining a bijection on alphabets.

Definition 12 *The **reverse** of the content $\alpha = (\alpha_1, \ldots, \alpha_M)$ is the tuple*

$$r(\alpha) = (\alpha_M, \alpha_{M-1}, \ldots, \alpha_1).$$

*In terms of alphabets, let A be a finite multiset of positive integers with maximum element M. The **reverse** of A, denoted $\overline{A}$, is the multiset consisting of the elements $M + 1 - a$ for all $a \in A$.*

We now can define bijections

$$\text{invcode} : (\mathcal{F}^\alpha_{(1^n)}; \text{inv}) \to (C_{(1^n),A}; \Sigma); \qquad \text{majcode} : (\mathcal{F}^{r(\alpha)}_{(n)}; \text{maj}) \to (C_{(1^n),A}; \Sigma).$$

Definition 13 *Let w be a word (corresponding to a filling of a row) consisting of the letters in the ordered alphabet $A = a_1 \leq \cdots \leq a_n$, with ties among the letters broken in the order they appear in w. We define $\text{invcode}(w) = c_1 \cdots c_n$ where c_i is the number of inversions having a_i as the smaller entry.*

For example, the inversion code of the filling

$$\boxed{3}\boxed{2}\boxed{4}\boxed{1}\boxed{3}\boxed{2}$$

is 313010, since the 1 is the smaller entry of 3 inversions, the first 2 is the smaller entry of 1 inversion, the second 2 is the smaller entry of 3 inversions, and so on. It will follow from Theorem 2 that:

Proposition 1 *The map invcode is an isomorphism of weighted sets $\text{invcode} : \mathcal{F}^\alpha_{(1^n)} \to C_{(1^n),A}$.*

To define the map majcode, we first require a standardization rule for fillings of columns.

Definition 14 *Let σ be any filling of a column of height n with positive integers. We define the **standardization labeling** on repeated entries as follows. Let i be a letter that occurs k times in σ.*

1. *Remove all entries larger than i to form a smaller column σ'.*

2. *Assign a label of k to the bottommost i that is either at the very bottom of σ' or for which the entries a and b directly north and south of it satisfy $a > b$. Repeat this process, labeling the next i by $k - 1$ and so on, until there are no i's that satisfy this condition.*

3. *Remove and label the remaining i's from top to bottom, decreasing the label by 1 each time.*

We define $\text{Standardize}(\sigma)$ to be the unique column filling using labels $1, 2, \ldots, n$ that respects the ordering of the entries of σ and breaks ties according to the standardization labeling.

Proposition 2 *For any column filling σ with alphabet A, let $\rho = \text{Standardize}(\sigma)$. Then ρ and σ have the same major index, and $\text{majcode}(\rho)$ is A-weakly increasing.*

We now can define the map majcode on words, that is, for one-column fillings.

Definition 15 *Let σ be any filling of a column shape $\mu = (1^r)$. We define*

$$\text{majcode}(\sigma) = \text{majcode}(\text{Standardize}(\sigma)),$$

where majcode of a standard filling is defined to be the majcode of its reading word.

Proposition 3 *The map majcode is a weighted set isomorphism $\mathcal{F}^{r(\alpha)}_{(1^n)} \to C_{(1^n),A}$ for any alphabet A with content α, and any one-column partition shape (1^n).*

See the full paper [5] for proofs of Propositions 1, 2, and 3.

3 Hook Shapes

We now demonstrate a new bijective proof of Conjecture 1 in the case that μ is a *hook shape*, that is, $\mu = (m, 1, 1, 1, \ldots, 1)$ for some m. The symmetry of inv and maj was demonstrated for fillings of hook shapes having *distinct* entries in [2], and makes use of the Foata bijection. In this section, we instead give a generalization the Carlitz bijection to this setting, which will hold for arbitrary fillings by the results in Section 2.1.

Definition 16 *Let σ be a filling of a hook shape μ. We define the* **hook codes** *of σ to be the pair of codes consisting of the* $\mathrm{invcode}$ *of its bottom row and the* $\mathrm{majcode}$ *of its leftmost column, along with the data of which entries occur in the row and which occur in the column.*

To extend the standardization orderings on the row and column of μ as defined in Section 2.1, if the corner square in μ is one of the repeated letters a of the filling, then we consider it the largest a in its column and the smallest a in its row. Thus we can define a standardization ordering on fillings of hook shapes: we order the letters smallest to largest, with the following tie-breaking rules.

- If two copies of the letter a appear in the (leftmost) column, the tie is broken as in Definition 14.

- If they appear in the (bottom) row, then the leftmost a comes first.

- If one appears in the column and the other in the row, the a in the column comes first.

This enables us to represent hook codes as a table, as shown in the following example.

Example 4 *Consider the filling σ of a hook shape shown below. The 2 in the corner is considered to be greater than the 2 above it and less than the 2 to its right. To represent the hook code of σ, we write the entries of the filling in the standardization ordering, and write the* $\mathrm{invcode}$ *and (the reverse of)* $\mathrm{majcode}$ *of the bottom row and left column respectively underneath the corresponding letters.*

	1	2	2	2	3	4	5	5
invcode			0	2	1		0	0
majcode	0	1	0			2		

Notice that the $\mathrm{majcode}$ *is written backwards, because the entries are in increasing order.*

We can now define our bijection.

Definition 17 *For any hook shape μ and content α, let $\phi : \mathcal{F}_\mu^\alpha \to \mathcal{F}_{\mu^*}^{r(\alpha)}$ be the map defined by interchanging the pair of hook codes of a filling and writing them backwards, and also reversing its alphabet.*

Example 5 *Starting with the tableau in Example 4, if we reverse the alphabet, interchange* $\mathrm{invcode}$ *and* $\mathrm{majcode}$*, and write the codes in backwards order, then we obtain the filling and pair of codes below. It follows that the filling in Example 4 maps to the filling below under ϕ.*

$$
\begin{array}{|c|}
\hline 4 \\
\hline 3 \\
\hline 1 \\
\hline 1 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|}
\hline 4 & 5 & 2 & 4 \\
\hline
\end{array}
$$

	1	1	2	3	4	4	4	5
invcode			2			0	1	0
majcode	0	0			1	2	0	

In [5] we show that ϕ is indeed a weight-preserving bijection on hook shapes:

Theorem 1 *We have that*

$$
\mathrm{maj}(\phi(\sigma)) = \mathrm{inv}(\sigma) \qquad and \qquad \mathrm{inv}(\phi(\sigma)) = \mathrm{maj}(\sigma)
$$

for any filling σ of a given hook shape μ. Moreover, ϕ is a bijection from $\mathcal{F}_\mu^\alpha$ to $\mathcal{F}_{\mu^*}^{r(\alpha)}$ for any content α.

4   Hall-Littlewood Specialization at $q = 0$

We now turn to the specialization in which one of the statistics is zero. In particular, setting $q = 0$, the symmetry relation becomes $\widetilde{H}_\mu(x; 0, t) = \widetilde{H}_{\mu^*}(x; t, 0)$, which is a symmetry relation on the transformed *Hall-Littlewood polynomials* $\widetilde{H}_\mu(x; t) := \widetilde{H}_\mu(x; 0, t)$. In this case the identity becomes

$$
\sum_{\substack{\sigma : \mu \to \mathbb{Z}_+ \\ \mathrm{inv}(\sigma) = 0}} t^{\mathrm{maj}(\sigma)} x^\sigma = \sum_{\substack{\rho : \mu^* \to \mathbb{Z}_+ \\ \mathrm{maj}(\rho) = 0}} t^{\mathrm{inv}(\rho)} x^\rho. \tag{3}
$$

Combinatorially, we wish to find natural morphisms

$$
\varphi : \mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0} \ \to \ \mathcal{F}_{\mu^*}^{r(\alpha)}|_{\mathrm{maj}=0}
$$

of weighted sets, where $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$ is equipped with the maj statistic, and $\mathcal{F}_{\mu^*}^\alpha|_{\mathrm{maj}=0}$ is equipped with the inv statistic. For the bijection $r(\alpha)$, we will use the *reverse* map of Definition 12.

In the case that $\mathrm{inv}(\sigma) = 0$, the statistic maj is essentially the same as the ***cocharge*** statistic defined by Lascoux and Schützenberger in [12]. This connection lies in the *cocharge word* construction (Figure 2). One of the key steps towards understanding this is the following:

Proposition 4 *Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition. Given a tuple of multisets $(A_1, \ldots, A_k)$ of positive integers where $|A_i| = \mu_i$ for all i, there is a unique filling σ of μ with $\mathrm{inv}(\sigma) = 0$ whose ith row contains precisely the numbers in A_i for all i.*

Proof: Since $\mathrm{inv}(\sigma) = 0$, the bottom row has the elements of A_1 in increasing order from left to right. We now induct on the rows. Suppose row i is filled in with entries $b_1, \ldots, b_r$ left to right. The leftmost entry a_1 of row $i+1$ must be the smallest element of A_{i+1} that comes after b_1 in "cyclic order" (i.e. either larger than b_1, or $\min A_{i+1}$ if $b_1 > \max A_{i+1}$). Then the next entry a_2 must be the smallest element of $A_{i+1} \setminus \{a_1\}$, that comes after b_2 in cyclic order, and so on. This uniquely determines row $i+1$. □

Definition 18 *The **cocharge word** of a filling $\sigma : \mu \to \mathbb{Z}_+$ is the word $\mathrm{cw}(\sigma) = i_1 i_2 \cdots i_n$ consisting of the row heights of the cells $u_k = (i_k, j_k)$, where $u_1, u_2, \ldots, u_n$ is the ordering of the cells of μ such that $\sigma(u_1) \geq \sigma(u_2) \geq \cdots \geq \sigma(u_n)$, with ties broken in reverse reading order.*

In [8], it was shown that for any filling $\sigma \in \mathcal{F}|_{\mathrm{inv}=0}$ we have

$$\mathrm{maj}(\sigma) = \mathrm{cocharge}(\mathrm{cw}(\sigma))$$

where $\mathrm{cocharge}$ is the statistic defined in [12]. It also turns out that any filling with no inv can be uniquely reconstructed from its cocharge word and its alphabet. This easily shown using the notion of a ***relative inversion*** defined in [5].

In what follows, we will only be considering the case $\alpha = (1^n)$, i.e. fillings having distinct entries $1, \ldots, n$. For the general case see [5].

4.1   Generalized Carlitz Codes

We now generalize Carlitz codes to fillings having inv or maj equal to 0, and content $\alpha = (1^n)$. Our generalization is motivated by the monomial basis of the Garsia-Procesi modules in [4], which are closely connected to the cocharge (maj) statistic.

Definition 19 *A word having letters in $\{0, 1, 2, \ldots\}$ is* **Yamanouchi** *if every suffix contains at least as many i's as $i + 1$'s for all $i \geq 0$. A word $w = w_1 \cdots w_n$ is μ-**sub-Yamanouchi**, or μ-**Carlitz**, if there exists a Yamanouchi word $v = v_1 \cdots v_n$ of content μ such that $w_i < v_i$ for all i.*

Example 6 *The sub-Yamanouchi words for $\mu = (1, 1, 1, \ldots, 1)$ are precisely the classical Carlitz codes.*

Definition 20 *We write C_μ for the set of all μ-sub-Yamanouchi codes, equipped with the sum statistic Σ.*

We now introduce the concept of the *monomial* of a code. The next three definitions are compatible with the notation in [4].

Definition 21 *Fix variables $x_1, x_2, \ldots$.. For any finite code c of length n, define its* **monomial** *to be*

$$x^c = x_n^{c_1} x_{n-1}^{c_2} \cdots x_1^{c_n}.$$

Also let $\mathcal{C}(\mu)$ be the set of all monomials x^c of μ-sub-Yamanouchi words c.

In [4], the authors define a similar set of monomials $\mathcal{B}(\mu)$, which are the generators of the S_n-modules R_μ whose Frobenius characteristics are precisely the Hall-Littlewood polynomials. In [5], we show that $\mathcal{C}(\mu) = \mathcal{B}(\mu)$, by showing that the sets $\mathcal{C}(\mu)$ satisfy the recursion in [4]. To state this recursion we require two more definitions, which follow the notation in [4].

Definition 22 *Given a partition μ, define $\mu^{(i)}$ to be the partition formed by removing the unique corner square whose row has the same length as the ith row μ_i.*

7	1	**4**		
5	**6**	1	2	
2	3	5	5	8

Fig. 2: A filling σ with cocharge word 132112311223. It is the unique filling with its given sets of row entries that has no inversions, by Proposition 4. This allows us to recover σ from its cocharge word and alphabet.

Definition 23 *Given a set of monomials $\mathcal{C}$ and a monomial m, we write $m \cdot \mathcal{C}$ to denote the set of all monomials of the form $m \cdot x$ where $x \in \mathcal{C}$.*

Definition 24 *The sets $\mathcal{B}(\mu)$ are defined by $\mathcal{B}((1)) = \{1\}$ and the recursion*

$$\mathcal{B}(\mu) = \bigsqcup_{i-1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{B}(\mu^{(i)}).$$

Proposition 5 *The sets $\mathcal{C}(\mu)$ satisfy the same recursion as $\mathcal{B}(\mu)$, and $\mathcal{C}_{\{1\}}((1)) = \{x_1\} = \mathcal{B}((1))$.*

This shows that $\mathcal{C}(\mu) = \mathcal{B}(\mu)$, as claimed.

4.2 Inversion Codes

We can now generalize the inversion code of a permutation to arbitrary fillings ρ with $\mathrm{maj}(\rho) = 0$.

Definition 25 *Let ρ be a filling of μ^* having $\mathrm{maj}(\rho) = 0$. Order its entries by size with ties broken in reading order to form a totally ordered alphabet $\{a_1, \ldots, a_n\}$. Then its **inversion code** is the sequence $\mathrm{invcode}(\rho) = c_1 \cdots c_n$ whose ith entry c_i is the number of attacking pairs having a_i as its smaller entry.*

Example 7 *There are three attacking pairs, as shown, in the filling below.*

Its inversion code is 0002100. Indeed, to compute the $\mathrm{invcode}$, we order the entries by size and reading order and record the number of attacking pairs having each entry as the smaller of the pair:

Entries	1	1	1	1	2	3	3
Code	0	0	0	2	1	0	0

Theorem 2 *The inversion code of any filling $\rho \in \mathcal{F}_{\mu^*}^\alpha$ is α-weakly increasing and μ-sub-Yamanouchi. Moreover, the map*

$$\mathrm{invcode} : \mathcal{F}_{\mu^*}^\alpha|_{\mathrm{maj}=0} \to C_{\mu, A}$$

is an isomorphism of weighted sets, where $C_{\mu, A}$ is the set of μ-sub-Yamanouchi words that are A-weakly increasing. In particular $\mathrm{invcode}$ restricts to a bijection $\mathcal{F}_{\mu^*}^{(1^n)}|_{\mathrm{maj}=0} \to C_\mu$.

See [5] for the full proof of this result.

4.3 Major Index Codes

To complete the proof of the Hall-Littlewood case with $\alpha = (1^n)$, it now suffices to find a weighted set isomorphism

$$\mathrm{majcode} : (\mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}; \mathrm{maj}) \to (C_\mu, \Sigma).$$

By Proposition 5, the μ-sub-Yamanouchi monomials satisfy the recursion $\mathcal{C}(\mu) = \bigsqcup_{i=1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{C}(\mu^{(i)})$. It now suffices to show that $\mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}$ satisfies a similar recursion, with its maj statistic recorded by the

exponents of the monomials. In particular, we first need an analog of the notion of "removing the largest entry" as in the Carlitz bijection. We wish to find an explicit weighted set isomorphism

$$\psi : (\mathcal{F}_\mu^{(1^n)}|_{\text{inv}=0}; \text{maj}) \to \bigsqcup_d (\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}|_{\text{inv}=0}; \text{maj} + d), \tag{4}$$

so that ψ sends an inversion-free filling T of μ to an inversion-free filling $\psi(T)$ of $\mu^{(d+1)}$ for some d, such that $\text{maj}(\psi(T)) = \text{maj}(T) - d$. Then if we define $\text{majcode}(T) = d_1 d_2 \ldots d_n$ where $d_k = \text{maj}(\psi^k(T)) - \text{maj}(\psi^{k-1}(T))$, we are done.

Such a map can be extracted from the work of Killpatrick [11], which gives a combinatorial proof of a recursion for a generating function involving ***charge***. The charge statistic, written ch, is related to the cocharge by the equation $\text{ch}(\mu) = n(\mu) - \text{cocharge}(\mu)$ where $n(\mu) = \sum_i (i-1) \cdot \mu_i$. Killpatrick defines W_μ to be the set of words of content μ, and lets $r_{i,\mu} = |\{j > i : \mu_j = \mu_i\}|$. The recursion is stated as:

$$\sum_{w \in W_\mu} q^{\text{ch}(w)} = \sum_i q^{r_{i,\mu}} \sum_{w \in W_{\mu^{(i)}}} q^{\text{ch}(w)}.$$

If we substitute $q \to 1/q$ and multiply both sides by $q^{n(\mu)}$, this becomes

$$\sum_{w \in W_\mu} q^{\text{cocharge}(w)} = \sum_i q^{i-1} \sum_{w \in W_{\mu^{(i)}}} q^{\text{cocharge}(w)},$$

which is essentially the same as the recursion for μ-sub-Yamanouchi codes. Killpatrick's work, when translated into the language of fillings via the cocharge word construction, gives a map ψ as in Equation (4). We now give a translated description of the map ψ.

Definition 26 *The* **crank** *of a filling σ having $\text{inv}(\sigma) = 0$ and alphabet $\{1, \ldots, n\}$ is the filling formed by (a) decreasing each entry $i \geq 2$ by 1 and replacing the entry 1 with n, and (b) rearranging the entries within each row in the unique way such that $\text{inv} = 0$, as in Proposition 4.*

Definition 27 *A* **crank orbit** *is the set of all fillings obtained by repeatedly applying the crank to a filling.*

Note that the crank orbits have sizes $|\mu|/d$ where d is some common divisor of the parts of μ. Furthermore, the crank orbits partition the fillings of μ into disjoint subsets. (See Figure 3.)

Let T be a filling with $\text{inv}(T) = 0$. We now give a five-step algorithm for computing $\psi(T)$, and throughout we use the example

$$T = \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}.$$

Step 1: List the crank orbits and mark the "special" fillings. List the entries of each crank orbit for μ in order by starting with a filling and repeatedly applying the crank. We define a ***special filling*** of such an orbit to be a filling for which the largest entry $n = |\mu|$ occurs in the bottom row of the tableau.

Step 2: Assign difference values plus one to the special fillings. For each special filling $\sigma^\star$, define

$$\text{diff}(\sigma^\star) = \text{maj}(\text{crank}^{-1}(\sigma^\star)) - \text{maj}(\sigma^\star)$$

Assign to each special filling the number

$$\ell(\sigma^\star) = \text{diff}(\sigma^\star) + 1.$$

$$
\begin{array}{cccc|cc}
\boxed{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} & \boxed{\begin{smallmatrix}4&1\\2&3\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&4\\1&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&3\\1&4\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&1\\2&4\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&4\\1&3\end{smallmatrix}}
\end{array}
$$

special?	$\star$			$\star$	$\star$	
maj	0	1	2	1	1	2
$\ell(\sigma^\star)$	2			2	2	
label	z_1	z_1	z_2	z_2	z_1	z_1

Fig. 3: The crank orbits for $\mu = (2,2)$, along with their special entries, maj and ℓ values, and labels.

The work in [11] shows that the values $\ell(\sigma^\star)$, ranging over all special fillings in a given orbit of size $|\mu|/d$, will be the sizes of the column lengths of the partition μ/d formed by dividing each of the rows of μ by d.

Step 3: Assign labelings. Starting with any special filling $\sigma^\star$ in a given orbit, label it z_1 and proceed to label each entry in the orbit according to the following algorithm. Continue labeling entries by z_1 in order until we have either labeled $\ell(\sigma_1^\star)$ of them, or until we encounter another special filling $\sigma_2^\star$. In either case, change our label to z_2 and start labeling entries with z_2 starting from $\sigma_2^\star$ in the same manner. If we finish labeling $\ell(\sigma_2^\star)$ fillings with a_2 and have not finished labeling with z_1's, return to z_1 until it is finished or we reach the next star, which we label z_3, and so on. (See Figure 3.) It turns out that we will end up with the same partition of the orbit given by the labels z_i no matter which special filling that we start at. [11]

Step 4: Sort the labeled entries into columns. For the crank orbit of T, sort all of the entries labeled z_1, all those labeled z_2, and so on each into their own column with the special entry at the bottom and the rest above it in the order they appear in the orbit. This forms a set of columns which, if arranged in decreasing order of height, forms the partition shape μ/d.

For example, for the first orbit in Figure 3, we have two columns:

$$
\begin{array}{cc}
\boxed{\begin{smallmatrix}4&1\\2&3\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&4\\1&2\end{smallmatrix}} \\[1em]
\boxed{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&3\\1&4\end{smallmatrix}}
\end{array}
$$

Step 5: Bumping from the bottom. Find the location of T in one of the columns produced in Step 4. Let i be the number of fillings below T in its column. Let $T^\star$ be the special filling in the bottom row of the column of T. Let cell $c = (\text{row}, \text{col})$ be the corner of $T^\star$ that is removed to form shape $\mu^{(i)}$, and let $T_\rightarrow^\star$ be the filling formed by deleting columns $1, \ldots, \text{col} - 1$ of $T^\star$. So, in the running example, T is in the upper right corner of the diagram from Step 4, and so $c = 2$, $i = 1$, $T^\star = \boxed{\begin{smallmatrix}2&3\\1&4\end{smallmatrix}}$ and $T_\rightarrow^\star = \boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}$.

Define the ***bumping sequence*** to be the sequence of entries $b_{\text{row}}, b_{\text{row}-1}, \ldots, b_1$ where b_{row} is the entry in square c, and for all j, b_j is the entry in row j of $T_\rightarrow^\star$ which is the largest entry less than b_{j+1} (or the very largest entry, if b_{j+1} is less than all entries in row j). It turns out that we always have $b_1 = n$.

Finally, remove $b_1 = n$ and bump down $b_1, \ldots, b_{\text{row}}$ one row each. Re-order the rows so that there are no inversions, and the resulting tableau is $\psi(T)$. In the running example we have $\psi(T) = \boxed{\begin{smallmatrix}2&\\1&3\end{smallmatrix}}$. Killpatrick's proof shows that $\text{maj}(\psi(T)) = \text{maj}(T) - i$. It follows that ψ gives rise to a map majcode that completes the proof of symmetry in this case.

Theorem 3 *In the case $\alpha = (1^n)$ of fillings with distinct entries, we have that $\varphi = \text{majcode}^{-1} \circ \text{invcode}$*

is an isomorphism of weighted sets

$$\varphi : \mathcal{F}_\mu^{(1^n)}|_{\mathrm{maj}=0} \to \mathcal{F}_{\mu^*}^{(1^n)}|_{\mathrm{inv}=0}.$$

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