FPSAC 2016 Vancouver, Canada DMTCS proc. **BC**, 2016, 803–814

Tropical Ideals

Diane Maclagan[1†] and Felipe Rincón[2‡]

[1] *Mathematics Institute, University of Warwick, United Kingdom.*
[2] *Department of Mathematics, University of Oslo, Norway.*

Abstract. We introduce and study a special class of ideals over the semiring of tropical polynomials, which we call tropical ideals, with the goal of developing a useful and solid algebraic foundation for tropical geometry. We explore their rich combinatorial structure, and prove that they satisfy numerous properties analogous to classical ideals.

Résumé. Nous introduisons et étudions une classe particulière d'idéals sur le demi-anneau des polynômes tropicaux, que nous appelons idéals tropicaux, dans la perspective de développer des fondations solides et utiles pour la géométrie tropicale. Nous explorons leur riche structure combinatoire, et nous prouvons qu'ils satisfont de nombreuses propriétés analogues à celles des idéaux classiques.

Keywords. tropical geometry, matroid, valuated matroid, tropical variety, Hilbert function, Gröbner complex, Nullstellensatz

1 Introduction

Tropical algebraic geometry is a piecewise linear shadow of algebraic geometry, in which varieties are replaced by certain polyhedral complexes that can be studied combinatorially. This area has grown significantly in the past decade and has had great success in numerous applications, like Mikhalkin's calculation of Gromov-Witten invariants of $\mathbb{P}^2$ [Mik05].

One current limitation of the theory, however, is that almost all techniques developed to date are focused on tropical varieties and tropical cycles, as opposed to schemes or more general spaces. Many of the standard algebraic tools of modern algebraic geometry thus do not yet have a tropical counterpart. These include elementary parts of algebraic geometry, such as uniqueness of irreducible decomposition. For instance, Figure 1 illustrates a tropical variety that admits two different "tropical irreducible decompositions". The current tools of tropical geometry do not allow us to distinguish between these two cases.

In [GG], Jeff and Noah Giansiracusa described a way of tropicalizing a subscheme of a toric variety using certain quotients of the semiring of tropical polynomials. The authors of this paper developed this further in [MR], clarifying to a greater extent the connection to tropical linear spaces and valuated matroids.

In this paper we generalize this point of view and study *tropical ideals*, which are ideals over the semiring of tropical polynomials whose homogeneous parts are 'matroidal'. Tropical ideals have a deep and interesting combinatorial structure, and we believe that they deserve a lot more study.

†Email: `D.Maclagan@warwick.ac.uk`
‡Email: `feliperi@math.uio.no`

We denote by $\overline{\mathbb{R}}$ the tropical semiring $\mathbb{R} \cup \{\infty\}$ with the operations tropical sum $\oplus = \min$ and tropical multiplication $\odot = +$. The semiring of tropical polynomials $\overline{\mathbb{R}}[\mathbf{x}] = \overline{\mathbb{R}}[x_0, \ldots, x_n]$ consists of polynomials with coefficients in $\overline{\mathbb{R}}$ where all operations are tropical. For simplicity, we describe our results in the fundamental case of homogeneous ideals. A more extensive treatment can be found in Section 2.

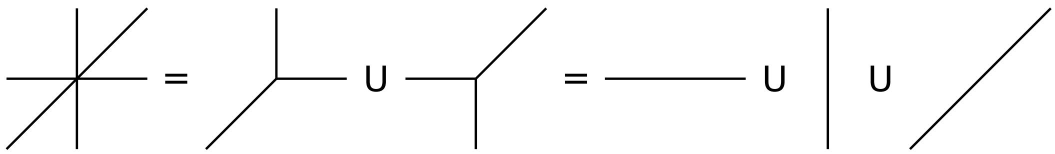

Fig. 1: Two tropical irreducible decompositions.

Definition 1.1 *A homogeneous ideal $J \subset \overline{\mathbb{R}}[\mathbf{x}]$ is a* tropical ideal *if for each degree $d \geq 0$ its homogeneous part J_d is the set of vectors of a valuated matroid, or equivalently, J_d is a tropical linear space. Concretely, J is a tropical ideal if it satisfies the following "monomial elimination axiom":*

- *For any $f, g \in J_d$ and any degree-d monomial $\mathbf{x}^{\mathbf{u}}$ for which $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}} \neq \infty$, there exists $h \in J_d$ such that $[h]_{\mathbf{x}^{\mathbf{u}}} = \infty$ and $[h]_{\mathbf{x}^{\mathbf{v}}} \geq \min([f]_{\mathbf{x}^{\mathbf{v}}}, [g]_{\mathbf{x}^{\mathbf{v}}})$ for any degree-d monomial $\mathbf{x}^{\mathbf{v}}$, with the equality holding whenever $[f]_{\mathbf{x}^{\mathbf{v}}} \neq [g]_{\mathbf{x}^{\mathbf{v}}}$.*

Here we use the notation $[f]_{\mathbf{x}^{\mathbf{u}}}$ to denote the coefficient of the monomial $\mathbf{x}^{\mathbf{u}}$ in the tropical polynomial f.

If K is a field equipped with a non-archimedean valuation, any classical polynomial $f \in K[\mathbf{x}]$ gives rise to a tropical polynomial $\operatorname{trop}(f) \in \overline{\mathbb{R}}[\mathbf{x}]$ by interpreting all operations tropically and replacing all coefficients by their valuation. If $I \subset K[\mathbf{x}]$ is a classical homogeneous ideal then the ideal

$$\operatorname{trop}(I) := \langle \operatorname{trop}(f) \mid f \in I \rangle \subset \overline{\mathbb{R}}[\mathbf{x}]$$

is a homogeneous tropical ideal in $\overline{\mathbb{R}}[\mathbf{x}]$. However, the class of tropical ideals is larger: we exhibit in Example 2.4 a tropical ideal that cannot be realized as $\operatorname{trop}(I)$ for any ideal $I \subset K[\mathbf{x}]$ over any field K.

As we describe below, the "monomial elimination axiom" required for tropical ideals makes up for the lack of additive inverses in the tropical semiring, and gives tropical ideals a rich algebraic structure reminiscent of classical ideals. In particular, tropical ideals seem to reflect the underlying geometry much better than general ideals of $\overline{\mathbb{R}}[\mathbf{x}]$; see Example 2.6.

Given any tropical polynomial $f \in \overline{\mathbb{R}}[\mathbf{x}]$, its corresponding hypersurface is defined as

$$V(f) := \{\mathbf{w} \in \overline{\mathbb{R}}^{n+1} : \text{the minimum in } f(\mathbf{w}) \text{ is achieved at least twice}\}.$$

For any ideal $J \subset \overline{\mathbb{R}}[\mathbf{x}]$, its corresponding *variety* is

$$V(J) := \bigcap_{f \in J} V(f).$$

If J is an arbitrary ideal in $\overline{\mathbb{R}}[\mathbf{x}]$ then $V(J)$ can be a fairly arbitrary subset of $\overline{\mathbb{R}}^{n+1}$; see Example 4.1. In particular, $V(J)$ might not even be polyhedral. However, if J is a tropical ideal, one of our main results shows that this is not the case:

Theorem 1.2 *If $J \subset \overline{\mathbb{R}}[\mathbf{x}]$ is a tropical ideal then the variety $V(J)$ in $\mathbb{R}^{n+1}$ is a finite polyhedral complex.*

Our proof of Theorem 1.2 generalizes the case where $J = \operatorname{trop}(I)$ for a classical ideal I: we develop a Gröbner theory for tropical ideals, and show that any tropical ideal has a finite $\mathbb{R}$-rational Gröbner complex, as in [MS15, §2.5]. The variety of J is then a subcomplex of its Gröbner complex.

A *tropical basis* for a tropical ideal J is a collection of polynomials in J that cut out the variety $V(J)$. Even though tropical ideals are generally not finitely generated, we show in Theorem 4.5 that they all admit a finite tropical basis.

We also investigate more algebraically flavored properties of tropical ideals. The fact that homogeneous parts of tropical ideals are tropical linear spaces allows us to naturally define the Hilbert function of any homogeneous tropical ideal. In the case where $J = \operatorname{trop}(I)$ for a classical ideal I, the Hilbert function of J agrees with the Hilbert function of I. In Corollary 3.8 we show that, just as in the classical case, the Hilbert function of any tropical ideal is eventually polynomial.

The semiring $\overline{\mathbb{R}}[\mathbf{x}]$ is not Noetherian, and tropical ideals are almost never finitely generated. Example 3.11 gives an infinite family of distinct tropical ideals $\{J^j\}_{j\geq 1}$, all of them having the same Hilbert function, such that for any $d \geq 0$, if $k, l \geq d$ then the tropical ideals J^k and J^l agree on all their homogeneous parts of degree at most d. However, we show that tropical ideals do satisfy the following Noetherian property.

Theorem 1.3 *(Ascending chain condition.) There is no infinite ascending chain $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ of tropical ideals.*

There are several versions of the Nullstellensatz for tropical geometry that can be found in the literature. Most of these results are about arbitrary finitely generated ideals in $\overline{\mathbb{R}}[\mathbf{x}]$. In our case, the rich structure we impose on tropical ideals allows us to use the results in [GP] to get the following elegant formulation.

Theorem 1.4 *(Tropical Nullstellensatz.) If $J \subset \overline{\mathbb{R}}[\mathbf{x}]$ is a tropical ideal then the variety $V(J) \cap \mathbb{R}^{n+1}$ in the tropical torus is empty if and only if J contains a monomial.*

Since all of our arguments for tropical ideals are of combinatorial nature, our approach has the appealing feature of providing completely combinatorial proofs for some well-known statements, like the existence of a (finite) Gröbner complex for any ideal $I \subset K[\mathbf{x}]$, Theorem 1.2 for ideals of the form $\operatorname{trop}(I)$ with $I \subset K[\mathbf{x}]$, and the classical version of Theorem 1.4.

2 Tropical Ideals

In this section we give the definition of tropical ideals, together with several examples. For this purpose, we first recall some of the basics of valuated matroids.

Throughout this paper we denote by $\overline{\mathbb{R}}$ the tropical semiring (or min-plus algebra)

$$\overline{\mathbb{R}} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot), \text{ where } \oplus := \min \text{ and } \odot := +.$$

We denote by

$$\overline{\mathbb{R}}[\mathbf{x}] = \overline{\mathbb{R}}[x_0, \ldots, x_n]$$

the semiring of tropical polynomials in the variables $\mathbf{x} = (x_0, \ldots, x_n)$. We write tropical monomials using the same notation as for classical monomials; for instance, we write x^2y^3 for the tropical monomial $x \odot x \odot y \odot y \odot y$. Elements of $\overline{\mathbb{R}}[\mathbf{x}]$ are polynomials with coefficients in $\overline{\mathbb{R}}$ where all operations are to be interpreted tropically. Explicitly, if $f \in \overline{\mathbb{R}}[\mathbf{x}]$ then f has the form $f(\mathbf{x}) = \bigoplus_{\mathbf{u}\in\mathbb{N}^{n+1}}(a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}})$, where $a_{\mathbf{u}} \in \overline{\mathbb{R}}$ and all but finitely many of the $a_{\mathbf{u}}$ equal ∞. Note that elements of $\overline{\mathbb{R}}[\mathbf{x}]$ are regarded as tropical polynomials and not functions; for example, the tropical polynomials $f(x) = x^2 \oplus 0$ and $g(x) = x^2 \oplus 1 \odot x \oplus 0$ are distinct, even though $f(w) = g(w)$ for all $w \in \mathbb{R}$. The *support* of a tropical polynomial $f = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ is

$$\operatorname{supp}(f) := \{\mathbf{u} \in \mathbb{N}^{n+1} : a_{\mathbf{u}} \neq \infty\}.$$

We call $a_{\mathbf{u}}$ the coefficient in f of the monomial $\mathbf{x}^{\mathbf{u}}$.

Valuated matroids are a generalization of the notion of matroids, introduced by Dress and Wenzel in [DW92]. In the tropical literature they are also known by the name of *tropical Plücker vectors* [SS04]. We now recall some of the necessary background on valuated matroids and tropical linear spaces; for basics of standard matroids, see, for example, [Oxl92].

Let E be a finite set, and let $r \in \mathbb{N}$. Denote by $\binom{E}{r}$ the collection of subsets of E of size r. A *valuated matroid* on the ground set E is a pair $\mathcal{M} = (E, p)$ where $p\colon \binom{E}{r} \to \overline{\mathbb{R}}$ satisfies the following properties:

- There exists $B \in \binom{E}{r}$ such that $p(B) \neq \infty$.
- *Tropical Plücker relations:* For every $A, B \in \binom{E}{r}$ and every $a \in A \setminus B$ there exists $b \in B \setminus A$ with

$$p(A) + p(B) \geq p(A \cup b - a) + p(B \cup a - b).$$

This version of the tropical Plücker relations can be easily seen to be equivalent to their more standard formulation used in the tropical literature (see, for example, [MS15, §4.4]). If $\mathcal{M} = (E, p)$ is a valuated matroid, its support

$$\operatorname{supp}(p) := \{B \in \tbinom{E}{r} : p(B) \neq \infty\}$$

is the collection of bases of a rank r matroid on the ground set E, called the *underlying matroid* $\underline{\mathcal{M}}$ of $\mathcal{M}$. The function p is called the *basis valuation function* of $\mathcal{M}$.

Just as ordinary matroids, valuated matroids have several different "cryptomorphic" definitions, some of which we now recall. For more information see [MT01].

Let $\mathcal{M}$ be a valuated matroid on the ground set E with basis valuation function $p : \binom{E}{r} \to \overline{\mathbb{R}}$. Given a basis B of $\underline{\mathcal{M}}$ and an element $e \in E \setminus B$, the (valuated) *fundamental circuit* $H(B, e)$ of $\mathcal{M}$ is the vector in $\overline{\mathbb{R}}^E$ defined by

$$H(B, e)_f := p(B \cup e - f) - p(B) \in \overline{\mathbb{R}} \qquad \text{for any } f \in E,$$

where we follow the convention that $p(B') = \infty$ if $|B'| > r$, and $\infty - a = \infty$ for any $a \in \mathbb{R}$. A (valuated) *circuit* of $\mathcal{M}$ is any vector in $\overline{\mathbb{R}}^E$ of the form $\lambda \odot H(B, e)$, where B is a basis of $\underline{\mathcal{M}}$, $e \in E \setminus B$, and $\lambda \in \mathbb{R}$. We denote by $\mathcal{C}(\mathcal{M})$ the collection of all circuits of $\mathcal{M}$. For any $H \in \overline{\mathbb{R}}^E$, its support is defined as

$$\operatorname{supp}(H) := \{e \in E : H_e \neq \infty\}.$$

The set of supports of the circuits of $\mathcal{M}$ is equal to the set of circuits of the underlying matroid $\underline{\mathcal{M}}$. Furthermore, if two circuits G and H of $\mathcal{M}$ have the same support then there exists $\lambda \in \mathbb{R}$ such that $G = \lambda \odot H$.

Collections of circuits of valuated matroids can be intrinsically characterized by a few axioms that generalize the classical circuit axioms for matroids (see [MT01]). The most important one of them is the following valuated elimination property.

- *Circuit elimination axiom:* For any $G, H \in \mathcal{C}(\mathcal{M})$ and any $e, f \in E$ such that $G_e = H_e \neq \infty$ and $G_f < H_f$, there exists $F \in \mathcal{C}(\mathcal{M})$ satisfying $F_e = \infty$, $F_f = G_f$, and $F \geq G \oplus H$.

Here we follow the convention that $F \geq G$ if and only if $F_e \geq G_e$ for all e.

A *cycle* of a classical matroid is a union of circuits. Its valuated counterpart is called a *vector*, which is defined as any point in the tropical convex hull of the valuated circuits. More explicitly, the set $\mathcal{V}(\mathcal{M})$ of vectors of $\mathcal{M}$ is equal to $\{\bigoplus_{F \in \mathcal{C}(\mathcal{M})} \lambda_F \odot F : \lambda_F \in \overline{\mathbb{R}} \text{ for all } F\}$. Vectors can also be characterized by just a few axioms, of which the following one is the most important.

- *Vector elimination axiom:* For any $G, H \in \mathcal{V}(\mathcal{M})$ and any $e \in E$ such that $G_e = H_e \neq \infty$, there exists $F \in \mathcal{V}(\mathcal{M})$ satisfying $F_e = \infty$, $F \geq G \oplus H$, and $F_f = G_f \oplus H_f$ for all $f \in E$ such that $G_f \neq H_f$.

The set of vectors of a valuated matroid is called a *tropical linear space* in the tropical literature. In the terminology used in [MS15, §4.4], if p is the basis valuation function of a valuated matroid $\mathcal{M}$ then $\mathcal{V}(\mathcal{M})$ is the tropical linear space $L_{p^\perp}$, where $p^\perp$ is the dual tropical Plücker vector $p^\perp(B) := p(E \backslash B)$. Tropical linear spaces are pure dimensional polyhedral complexes in $\mathbb{R}^E$, of dimension equal to $\dim(\mathcal{V}(\mathcal{M})) = |E| - r(\mathcal{M})$ (see [Spe08]).

We now introduce the main object of study of our paper. Let Mon_d be the set of monomials of degree d in the variables $\mathbf{x} = (x_0, \ldots, x_n)$. We will identify elements of $\overline{\mathbb{R}}^{\mathrm{Mon}_d}$ with homogeneous tropical polynomials of degree d. In this way, if $\mathcal{M}$ is a valuated matroid on the ground set Mon_d, circuits and vectors of $\mathcal{M}$ can be thought of as tropical polynomials in $\overline{\mathbb{R}}[\mathbf{x}]_d$.

Definition 2.1 *(*Homogeneous tropical ideals.*) A homogeneous ideal $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ is a* tropical ideal *if for each $d \geq 0$ its degree-d part I_d is the collection of vectors of a valuated matroid $\mathcal{M}_d$ on Mon_d, i.e., it is a tropical linear space in $\overline{\mathbb{R}}^{\mathrm{Mon}_d}$. Equivalently, I is a homogeneous tropical ideal if it can be written as a direct sum $I = \bigoplus_{d \geq 0} \mathcal{V}(\mathcal{M}_d)$ for some sequence of valuated matroids $(\mathcal{M}_d)_{d \geq 0}$, where the ground set of $\mathcal{M}_d$ is Mon_d.*

If I is a homogeneous tropical ideal, we will denote by $\mathcal{M}_d(I)$ the valuated matroid such that $I_d = \mathcal{V}(\mathcal{M}_d(I))$.

This definition agrees with Definition 1.1 in the introduction, since the axioms for vectors other than the vector elimination axiom are automatically satisfied by the homogeneous parts of any ideal $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ (see [MT01]).

Not every homogeneous ideal in $\overline{\mathbb{R}}[\mathbf{x}]$ is a tropical ideal. As an example, consider the ideal I in $\overline{\mathbb{R}}[x, y]$ generated by $x \oplus y$. The degree-two part of this ideal is the $\overline{\mathbb{R}}$-semimodule generated by $x^2 \oplus xy$, and $xy \oplus y^2$. This is not the set of vectors of a valuated matroid on $\mathrm{Mon}_2 = \{x^2, xy, y^2\}$, as its elements do not satisfy the vector elimination axiom: the polynomial $x^2 \oplus y^2$ would be required in I by the vector elimination axiom applied to the two generators.

Example 2.2 *(*Realizable tropical ideals.*) Let K be a field with a valuation map $\mathrm{val} : K \to \overline{\mathbb{R}}$. Any polynomial $f \in K[\mathbf{x}]$ gives rise to a tropical polynomial $\mathrm{trop}(f) \in \overline{\mathbb{R}}[\mathbf{x}]$ by interpreting all operations tropically and replacing any coefficients by their valuation; i.e., if $f = \sum c_{\mathbf{u}} \cdot \mathbf{x}^{\mathbf{u}}$, then $\mathrm{trop}(f) := \bigoplus \mathrm{val}(c_{\mathbf{u}}) \odot \mathbf{x}^{\mathbf{u}}$. If $J \subset K[\mathbf{x}]$ is any homogeneous ideal then the ideal*

$$\mathrm{trop}(J) := \langle \mathrm{trop}(f) \mid f \in J \rangle \subset \overline{\mathbb{R}}[\mathbf{x}]$$

is a homogeneous tropical ideal. A tropical ideal arising in this way is called realizable *(over the field K).*

A special class of realizable homogeneous tropical ideals consists of the tropical equivalent of the homogeneous ideal of a point in projective space. In Example 4.7 we will see that these are precisely the maximal homogeneous tropical ideals of $\overline{\mathbb{R}}[\mathbf{x}]$.

Example 2.3 *(*The homogeneous ideal of a point.*) Fix $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \overline{\mathbb{R}}^{n+1}$ with $\mathbf{a} \neq (\infty)^{n+1}$. Let $J_{\mathbf{a}}$ be the ideal generated by all homogeneous polynomials $f \in \overline{\mathbb{R}}[x_0, \ldots, x_n]$ for which the minimum in $f(\mathbf{a})$ is achieved at least twice. We claim that $J_{\mathbf{a}}$ is a tropical ideal. In addition, if K is a valued field with $a_i \in \operatorname{im} \operatorname{val}$ for all $0 \leq i \leq n$, then $J_{\mathbf{a}}$ is the tropicalization of* any *ideal J_α of a point $\alpha \in \mathbb{P}^n$ with $\operatorname{val}(\alpha_i) = a_i$.*

To prove the first claim, one can show that $J_{\mathbf{a}}$ is generated by the binomials of the form $(\mathbf{a} \cdot \mathbf{v}) \odot \mathbf{x}^{\mathbf{u}} \oplus (\mathbf{a} \cdot \mathbf{u}) \odot \mathbf{x}^{\mathbf{v}}$ with $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{x}^{\mathbf{v}})$, which satisfy the valuated circuit elimination axiom. We omit the details.

Suppose now that $\alpha \in \mathbb{P}^n$ satisfies $\operatorname{val}(\alpha) = \mathbf{a}$. The homogeneous ideal $J_\alpha \subset K[x_0, \ldots, x_n]$ of the point α contains the binomials $\alpha^{\mathbf{v}}\mathbf{x}^{\mathbf{u}} - \alpha^{\mathbf{u}}\mathbf{x}^{\mathbf{v}}$ for all pairs $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}$ with $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{x}^{\mathbf{v}})$, thus $J_{\mathbf{a}} \subseteq \operatorname{trop}(J_\alpha)$. If the inclusion were proper, there would be $h \in J_\alpha$ with $\operatorname{trop}(h) \notin J_{\mathbf{a}}$. This would contradict the fact that $\alpha \in V(h)$, as $\mathbf{a} = \operatorname{val}(\alpha) \in \operatorname{trop}(V(h)) = V(\operatorname{trop}(h))$.

We now present a tropical ideal that is not realizable over any field K.

Example 2.4 *(*A non-realizable tropical ideal.*) For any $n \geq 2$, we give an example of a homogeneous tropical ideal in $\overline{\mathbb{R}}[x_0, \ldots, x_n]$ that is not realizable over any field. Its underlying variety is, however, the standard tropical line in tropical projective space $\mathbb{TP}^n$.*

For $d \geq 0$, let $\mathcal{M}_d$ be the rank $d+1$ valuated matroid on the ground set Mon_d whose basis valuation function $p_d : \binom{\operatorname{Mon}_d}{d+1} \to \overline{\mathbb{R}}$ is given by

$$p_d(B) := \begin{cases} 0 & \text{if for any } k \leq d \text{ and any } \mathbf{x}^{\mathbf{v}} \in \operatorname{Mon}_k \text{ we have} \\ & \qquad |\{\mathbf{x}^{\mathbf{u}} \in B \mid \mathbf{x}^{\mathbf{v}} \text{ divides } \mathbf{x}^{\mathbf{u}}\}| \leq d - k + 1, \\ \infty & \text{otherwise.} \end{cases}$$

Geometrically, if we think of Mon_d as the set of lattice points inside a simplex of size d, the function p_d assigns the value 0 precisely to those $(d+1)$-subsets B of Mon_d such that for any $k \leq d$, the subset B contains at most $d - k + 1$ monomials from any simplex in Mon_d of size $d - k$. The fact that p_d is indeed the basis valuation function of a valuated matroid follows from [AB07, Theorem 4.1], where the corresponding underlying matroid is studied in detail.

The circuits of $\mathcal{M}_d$ are the tropical polynomials of the form $H = \lambda \odot \bigoplus_{\mathbf{u} \in C} \mathbf{x}^{\mathbf{u}}$ with $\lambda \in \mathbb{R}$ and C a minimal subset of Mon_d satisfying $|C| > d - \deg(\gcd(C)) + 1$. From this description it is clear that if H is a circuit of $\mathcal{M}_d$ and x_i is any variable then $x_i \odot H$ is a circuit of $\mathcal{M}_{d+1}$. It follows that $I = \bigoplus_{d \geq 0} \mathcal{V}(\mathcal{M}_d)$ is a tropical ideal.

To show that I is a non-realizable tropical ideal, note that the tropical polynomial $F = x_0 \oplus x_1 \oplus x_2$ is a circuit of $\mathcal{M}_1$, and in particular $F \in I$. If I is realizable, say $I = \operatorname{trop}(J)$ for some ideal $J \subset K[x_0, \ldots, x_n]$, then there exists some $f \in J$ such that $F = \operatorname{trop}(f)$. After a suitable scaling of the variables, we can assume that $f = x_0 + x_1 + x_2$. The polynomial

$$g := f \cdot (x_0^2 + x_1^2 + x_2^2 - x_0x_1 - x_0x_2 - x_1x_2) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2$$

is then a polynomial in J, and thus $G := \mathrm{trop}(g) \in I$*. However, this contradicts the fact that* $\mathrm{supp}(G) \subset \{x_0^3, x_1^3, x_2^3, x_0x_1x_2\}$ *is an independent set in the underlying matroid* $\underline{\mathcal{M}_3}$.

We now extend the definition of tropical ideals to cover non-homogeneous ideals.

Definition 2.5 *(*Non-homogeneous tropical ideals.*) The homogenization of a polynomial* $f = \bigoplus a_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}}$ *in* $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ *is*

$$\tilde{f} = \bigoplus a_{\mathbf{u}} \odot x_0^{d-|\mathbf{u}|} \odot \mathbf{x}^{\mathbf{u}} \in \overline{\mathbb{R}}[x_0, x_1, \ldots, x_n],$$

where $|\mathbf{u}| := \sum_{i=1}^n u_i$ *and* $d := \max_{a_{\mathbf{u}} \neq \infty} |\mathbf{u}|$*. The homogenization of an ideal* $I \subset \overline{\mathbb{R}}[x_1, \ldots, x_n]$ *is the homogeneous ideal* $\tilde{I} := \langle \tilde{f} : f \in I \rangle \subset \overline{\mathbb{R}}[x_0, x_1, \ldots, x_n]$.

An ideal $I \subset \overline{\mathbb{R}}[x_1, \ldots, x_n]$ *is a* tropical ideal *if its homogenization is a tropical ideal in* $\overline{\mathbb{R}}[x_0, \ldots, x_n]$.

An ideal $J \subset \overline{\mathbb{R}}[\mathbf{x}]$ is *prime* if $f \odot g \in J$ implies $f \in J$ or $g \in J$. Even in the semiring $\overline{\mathbb{R}}[x]$ of tropical polynomials in one variable, one can find uncountable chains of nested prime ideals (see [GG, Example 3.4]). However, considering only *tropical* prime ideals of $\overline{\mathbb{R}}[\mathbf{x}]$ seems to reflect the underlying geometry much better: if we restrict ourselves only to tropical prime ideals then the Krull dimension of $\overline{\mathbb{R}}[x]$ is 1, as explained below.

Example 2.6 *(*Tropical prime ideals of $\overline{\mathbb{R}}[x]$.*) We classify all tropical prime ideals in the semiring* $\overline{\mathbb{R}}[x]$ *of tropical polynomials in one variable. For any* $a \in \overline{\mathbb{R}}$*, let* J_a *be the ideal consisting of all tropical polynomials* F *whose bend locus* $V(F)$ *contains* a*. The polynomials of minimal support in* J_a *are (up to scaling) the binomials* $x^i \oplus a^{i-j} \odot x^j$ *with* $i > j$*, and any polynomial in* J_a *is a tropical sum of them. Since these binomials satisfy the circuit elimination axiom for valuated matroids,* J_a *is a tropical ideal. Moreover, the fact that any two tropical polynomials* F, G *satisfy* $V(F \odot G) = V(F) \cup V(G)$ *implies that* J_a *is a tropical prime ideal. We will prove that, in fact, these are the only proper tropical prime ideals in* $\overline{\mathbb{R}}[x]$*, together with the "zero ideal"* $\langle \infty \rangle$.

We say that a tropical polynomial $F(x) = \bigoplus_{i=0}^d c_i \odot x^i$ *is* convex *if its coefficients form a convex sequence, that is,* $c_i \leq \frac{1}{2}(c_{i-1} + c_{i+1})$ *for all* $0 < i < d$*. The product of two convex tropical polynomials is also convex. We will first show that any non-zero ideal* J *in* $\overline{\mathbb{R}}[x]$ *contains a convex polynomial. For this purpose, suppose* $G \in J$ *with* $G \neq \infty$*, and let* $\hat{G}$ *be its convexification: the coefficient* $\hat{c}_i$ *of the monomial* x^i *in* $\hat{G}$ *is the smallest number in* $\overline{\mathbb{R}}$ *such that* $G(b) = \hat{G}(b)$ *for all* $b \in \overline{\mathbb{R}}$*. Note that the polynomial* $\hat{G}$ *is indeed convex. One can prove that* $G \odot \hat{G} = \hat{G} \odot \hat{G}$*. Since* $\hat{G} \odot \hat{G}$ *is a product of convex polynomials, it is also convex. The polynomial* $G \odot \hat{G}$ *is thus a convex polynomial in the ideal* J.

Now, if J *is a non-zero tropical prime ideal in* $\overline{\mathbb{R}}[x]$*, we have shown that* J *contains a convex polynomial* F*. It is a well-known fact that convex polynomials in* $\overline{\mathbb{R}}[x]$ *factor into linear factors. Since* J *is prime,* J *must contain one of these linear factors, say* $(x \oplus a) \in J$*, with* $a \in \overline{\mathbb{R}}$*. Multiplying by powers of* x*, we see that* J *must contain all polynomials of the form* $x^i \oplus a \odot x^{i-1}$*. Furthermore, the monomial elimination axiom for tropical ideals forces* J *to contain all binomials* $x^i \oplus a^{i-j} \odot x^j$ *with* $i > j$*. Since these binomials generate the ideal* J_a*, we see that* J *must contain the whole ideal* J_a*. If* J *were strictly greater than* J_a*, it would contain a polynomial* F' *with* $V(F') \not\ni a$*. Repeating a similar argument,* J *would contain the convex polynomial* $\hat{F'} \odot \hat{F'}$*, which has the same tropical roots as* F' *(with double the multiplicity). Since* J *is prime,* J *would also contain one of its linear factors, say* $(x \oplus a') \in J$*, with* $a' \neq a$*. But then, the monomial elimination axiom applied to* $x \oplus a$ *and* $x \oplus a'$ *forces* J *to contain the constant* $\min(a, a')$*, which implies that* J *is the unit ideal.*

3 Gröbner theory for tropical ideals

In this section we develop a Gröbner theory for tropical ideals in $\overline{\mathbb{R}}[\mathbf{x}]$ and use it to prove some basic properties of tropical ideals. These include the eventual polynomiality of their Hilbert functions, and the fact that tropical ideals satisfy the ascending chain condition.

We denote by $\mathbb{B}$ the Boolean subsemiring of $\overline{\mathbb{R}}$

$$\mathbb{B} := (\{0, \infty\}, \oplus, \odot).$$

Initial ideals of tropical ideals in $\overline{\mathbb{R}}[\mathbf{x}]$ will be tropical ideals in the semiring of tropical polynomials $\mathbb{B}[\mathbf{x}]$, as we define below. We will sometimes identify subsets of Mon_d with homogeneous tropical polynomials in $\mathbb{B}[\mathbf{x}]$ of degree d, via the correspondence that takes any $S \subset \mathrm{Mon}_d$ to $\bigoplus_{\mathbf{u} \in S} \mathbf{x}^{\mathbf{u}} \in \mathbb{B}[\mathbf{x}]$. In this way, if M is an (ordinary) matroid on the ground set Mon_d, circuits and cycles (i.e., unions of circuits) of M can be thought of as tropical polynomials in $\mathbb{B}[\mathbf{x}]_d$.

Definition 3.1 *A homogeneous ideal $I \subset \mathbb{B}[\mathbf{x}]$ is a* tropical ideal *if for each $d \geq 0$ its degree-d part I_d is the collection of cycles (i.e., unions of circuits) of an ordinary matroid on Mon_d. If $I \subset \mathbb{B}[\mathbf{x}]$ is a tropical ideal, we will denote by $M_d(I)$ the matroid on Mon_d such that I_d is the collection of cycles of $M_d(I)$.*

We now define initial terms of tropical polynomials, although in a slightly more general context that will be useful later. If E is any finite set, $H \in \overline{\mathbb{R}}^E$, and $\mathbf{w} \in \mathbb{R}^E$, we let

$$\mathrm{in}_{\mathbf{w}} H := \{a \in E : H_a + w_a \text{ is minimal among all } a \in E\}$$

be the $\mathbf{w}$-*initial term* of H. By taking $E = \mathrm{Mon}_d$, this defines initial terms for tropical polynomials in $\overline{\mathbb{R}}[\mathbf{x}]$ and thus in $\mathbb{B}[\mathbf{x}]$. Note that the $\mathbf{w}$-initial term of a tropical polynomial is a collection of monomials, and thus it can be thought of as a polynomial in $\mathbb{B}[\mathbf{x}]$.

Definition 3.2 *(*Initial ideals.*) Let I be a homogeneous ideal in $\overline{\mathbb{R}}[\mathbf{x}]$, and let $\mathbf{w} \in \mathbb{R}^{n+1}$. The* initial ideal *$\mathrm{in}_{\mathbf{w}} I$ is the homogeneous ideal in $\mathbb{B}[\mathbf{x}]$ whose homogeneous parts are*

$$(\mathrm{in}_{\mathbf{w}} I)_d := \{\mathrm{in}_{\mathbf{w}_d} F \mid F \in I_d\},$$

where $\mathbf{w}_d : \mathrm{Mon}_d \to \mathbb{R}$ is defined as $\mathbf{w}_d(\mathbf{x}^{\mathbf{u}}) := \sum_{i=1}^n u_i w_i$. The fact that $\mathrm{in}_{\mathbf{w}} I$ is an ideal in $\mathbb{B}[\mathbf{x}]$ follows easily from the fact that I is an ideal in $\overline{\mathbb{R}}[\mathbf{x}]$. If I is an ideal in $\mathbb{B}[\mathbf{x}]$, we define its initial ideal *$\mathrm{in}_{\mathbf{w}} I$ to be equal to $\mathrm{in}_{\mathbf{w}} I'$, where I' is the ideal generated by I in $\overline{\mathbb{R}}[\mathbf{x}]$ under the natural inclusion $\mathbb{B}[\mathbf{x}] \hookrightarrow \overline{\mathbb{R}}[\mathbf{x}]$.*

Our definition of initial ideals is compatible with the usual definition of initial ideals used in tropical geometry, in the sense that for any homogeneous ideal $J \subset K[\mathbf{x}]$ we have $\mathrm{trop}(\mathrm{in}_{\mathbf{w}} J) = \mathrm{in}_{\mathbf{w}} \mathrm{trop}(J)$.

Our main result in this section will show that initial ideals of tropical ideals are also tropical ideals, which follows from the following key fact about valuated matroids. We omit its proof.

Lemma 3.3 *Let $\mathcal{M} = (E, p)$ be a rank r valuated matroid, and let $\mathbf{w} = (w_e)_{e \in E} \in \mathbb{R}^E$. Then*

$$\mathrm{in}_{\mathbf{w}} \mathcal{B}(\mathcal{M}) := \left\{ B \in \binom{E}{r} : p(B) - \textstyle\sum_{e \in B} w_e \text{ is minimal among all } B \in \binom{E}{r} \right\}$$

is the collection of bases of an (ordinary) matroid $\mathrm{in}_{\mathbf{w}} \mathcal{M}$ of rank r on the ground set E. Its circuits are the minimal elements *in the set $\mathrm{in}_{\mathbf{w}} \mathcal{C}(\mathcal{M}) := \{\mathrm{in}_{\mathbf{w}} H \subset E : H \in \mathcal{C}(\mathcal{M})\}$, and its set of cycles (i.e., unions of circuits) is*

$$\mathrm{in}_{\mathbf{w}} \mathcal{V}(\mathcal{M}) := \{\mathrm{in}_{\mathbf{w}} H \subset E : H \in \mathcal{V}(\mathcal{M})\}.$$

Lemma 3.3 implies the following description of initial tropical ideals.

Theorem 3.4 *Let I be a homogeneous tropical ideal in $\overline{\mathbb{R}}[\mathbf{x}]$, and let $\mathbf{w} \in \mathbb{R}^{n+1}$. Then the initial ideal $\operatorname{in}_{\mathbf{w}} I$ is a tropical ideal in $\mathbb{B}[\mathbf{x}]$, with associated matroids $M_d(\operatorname{in}_{\mathbf{w}} I) = \operatorname{in}_{\mathbf{w}} \mathcal{M}_d(I)$.*

Homogeneous tropical ideals have a well-defined Hilbert function.

Definition 3.5 *(*Hilbert functions.*) If $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ is a homogeneous tropical ideal, its* Hilbert function *is the function $H_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by*

$$H_I(d) := \textit{rank of } \mathcal{M}_d(I).$$

Similarly, the Hilbert function *of a tropical ideal $I \subset \mathbb{B}[\mathbf{x}]$ is the function $H_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by $H_I(d) :=$ rank of $M_d(I)$.*

Since tropicalization preserves dimensions of linear spaces, Hilbert functions are preserved under tropicalization: If $J \subset K[\mathbf{x}]$ is any homogeneous ideal then

$$H_{\operatorname{trop}(J)}(d) = \dim_K((K[\mathbf{x}]/J)_d)$$

for any $d \in \mathbb{Z}_{\geq 0}$.

Lemma 3.3 and Theorem 3.4 imply the following result.

Proposition 3.6 *Let I be a homogeneous tropical ideal in $\overline{\mathbb{R}}[\mathbf{x}]$ or $\mathbb{B}[\mathbf{x}]$, and let $\mathbf{w} \in \mathbb{R}^{n+1}$. Then for any $d \in \mathbb{Z}_{\geq 0}$ we have $H_I(d) = H_{\operatorname{in}_{\mathbf{w}} I}(d)$.*

The following result will be useful in our study of tropical ideals.

Proposition 3.7 *If I is a homogeneous tropical ideal in $\overline{\mathbb{R}}[\mathbf{x}]$ or $\mathbb{B}[\mathbf{x}]$ then there exists $\mathbf{w} \in \mathbb{R}^{n+1}$ such that $\operatorname{in}_{\mathbf{w}}(I)$ is generated by monomials. In fact, the set of $\mathbf{w}$ for which $\operatorname{in}_{\mathbf{w}}(I)$ is not generated by monomials has measure zero in $\mathbb{R}^{n+1}$.*

As an immediate application, we have the following result about Hilbert functions of tropical ideals.

Corollary 3.8 *(*Hilbert polynomial.*) If I is a homogeneous tropical ideal in $\overline{\mathbb{R}}[\mathbf{x}]$ or $\mathbb{B}[\mathbf{x}]$, then its Hilbert function H_I is eventually polynomial.*

Proof: Using Proposition 3.7 and Proposition 3.6, we can reduce to the case that I is a tropical ideal in $\mathbb{B}[\mathbf{x}]$ generated by monomials. In this case, the set of monomials in I determine its Hilbert function in the same way they do for classical ideals in $K[\mathbf{x}]$. The result then follows from the classical case. □

The following example shows that tropical ideals are typically not finitely generated. Nonetheless, we show that tropical ideals satisfy the ascending chain condition.

Example 3.9 *(*Tropical ideals are not finitely generated.*) Let $J = \langle x - y \rangle \subseteq K[x, y]$, and let $I = \operatorname{trop}(J) \subseteq \overline{\mathbb{R}}[x, y]$. Note that $x^d - y^d \in J$ for all $d \geq 1$, so $x^d \oplus y^d \in I$ for all $d \geq 1$. Suppose that $F_1, \dots, F_r$ is a finite generating set for I. Then for all $d \geq 1$ we can write $x^d \oplus y^d = \bigoplus H_{id} \odot F_i$. Since there is no cancellation in $\overline{\mathbb{R}}[x, y]$, if m is a monomial occurring in H_{id} for some d, and m' is a monomial occurring in F_i, then mm' occurs in $H_{id} \odot F_i$, and thus in $x^d \oplus y^d$. This is only possible if each F_i is either $x^d \oplus y^d$ or a power of x or y. Since there are only finitely many F_i, it follows that $\overline{\mathbb{R}}[x, y]_d = I_d$ for $d \gg 0$, and so the Hilbert function of I equals zero for $d \gg 0$. This contradicts the fact that the Hilbert functions of I and J agree, thus we conclude that I is not finitely generated.*

Proposition 3.10 *(*Ascending chain condition.*) If I, J are homogeneous tropical ideals in $\overline{\mathbb{R}}[\mathbf{x}]$ with $I \subseteq J$ and identical Hilbert functions then $I = J$. Furthermore, there is no infinite ascending chain $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ of homogeneous tropical ideals.*

Proof: Suppose $I \subseteq J$ are tropical ideals with the same Hilbert function. Then, for any degree d, we have an inclusion $I_d \subseteq J_d$ of tropical linear spaces of the same dimension. By [Rin12, Lemma 7.4], any such inclusion has to be an equality, thus $I_d = J_d$ for all d, and $I = J$.

Now, suppose that $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is an infinite ascending chain of tropical ideals. By Proposition 3.7, we can choose a $\mathbf{w} \in \mathbb{R}^{n+1}$ such that $\text{in}_{\mathbf{w}} I_j$ is generated by monomials for all j. The chain $\text{in}_{\mathbf{w}} I_1 \subseteq \text{in}_{\mathbf{w}} I_2 \subseteq \text{in}_{\mathbf{w}} I_3 \subseteq \ldots$ is then an ascending chain of monomial ideals, so by the classical theory it has to stabilize. Since $\text{in}_{\mathbf{w}} I_j$ and I_j have the same Hilbert functions, it follows that for large enough j the Hilbert functions of the tropical ideals I_j are all equal, which leads to a contradiction. □

Example 3.11 *(*Tropical ideals are not determined in low degree.*) We present an infinite family of homogeneous tropical ideals $(J'_m)_{m>0}$ in $\overline{\mathbb{R}}[x, y, z, w]$, all of them having the same Hilbert function, such that for any $d \geq 0$, if $k, l \geq d$ then the tropical ideals J'_k and J'_l agree on all their homogeneous parts of degree at most d, i.e., $(J'_k)_i = (J'_l)_i$ for all $i \leq d$. This implies that there is no bound D depending only on the Hilbert function of a tropical ideal I for which the homogeneous parts $(I_i)_{i \leq D}$ of degree at most D determine the whole tropical ideal I.*

For any real number $\lambda > 0$, consider the classical ideal $I_\lambda = \langle x - z - w,\ y - z - \lambda w \rangle$ in $\mathbb{C}[x, y, z, w]$, and let $J_\lambda := \text{trop}(I_\lambda)$. The Hilbert function of I_λ is the same as the one of J_λ, which gives $H_{J_\lambda}(d) = d + 1$. We claim that the tropical polynomial

$$F := x^n \oplus y \odot z^{n-1} \oplus z^{n-2} \odot w^2 \oplus z^{n-3} \odot w^3 \oplus \cdots \oplus z \odot w^{n-1} \oplus w^n$$

is in the tropical ideal J_λ if and only if $\lambda = n$. To prove our claim, note that $x^n - (z + w)^n \in I_\lambda$, since $x - (z + w) \in I_\lambda$. Thus the polynomial

$$\begin{aligned} f &:= x^n - (z + w)^n - z^{n-1}(y - z - \lambda w) \\ &= x^n - z^{n-1}y - (n - \lambda)z^{n-1}w - \binom{n}{2} z^{n-2}w^2 - \cdots - \binom{n}{n-1} zw^{n-1} - w^n \end{aligned}$$

is also in I_λ. If $\lambda = n$ then $\text{trop}(f) = F$, and so $F \in J_\lambda$. We omit the reverse implication due to space constraints. A similar analysis can be used to show that for any degree $d \geq 0$, the degree-d part of the tropical ideals I_λ are all equal except for finitely many values of λ. We call this the 'generic homogeneous part'.

We define our family of tropical ideals $(J'_m)_{m>0}$ inductively as follows. Let $N_0 = 1$, and suppose we have defined the tropical ideals J'_k for all $k < m$. Define $J'_m := J_{N_m}$, where N_m is a large enough integer such that for any degree $d \leq N_{m-1}$, the degree-d part of J_{N_m} is the generic homogeneous part. Such an N_m exists according to our discussion above. Since the matroid $M_{N_m}(J_{N_m})$ is not generic, it follows that $N_m > N_{m-1}$, and moreover, all the tropical ideals J_{N_m} defined in this way must be distinct. Also, by induction, we must have $N_{m-1} \geq m$. We conclude that the family $(J'_m)_{m>0}$ satisfies the required conditions: For any $d > 0$, if $k, l \geq d$ then the homogeneous parts of $J'_k = J_{N_k}$ and $J'_l = J_{N_l}$ of degree at most d are generic, since N_{k-1} and N_{l-1} are both greater than d.

4 Gröbner complex for tropical ideals

In this section we define and study the Gröbner complex of a tropical ideal, and use it to show that the underlying variety of any tropical ideal is always the support of a finite polyhedral complex. We start with an example that shows that this is not the case for arbitrary ideals of $\overline{\mathbb{R}}[\mathbf{x}]$.

Example 4.1 *Let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary collection of tropical polynomials in $\overline{\mathbb{R}}[\mathbf{x}]$, and consider the ideal $I = \langle F_\alpha \rangle_{\alpha \in A}$ in $\overline{\mathbb{R}}[\mathbf{x}]$ that they generate. Its variety $V(I)$ in $\overline{\mathbb{R}}^{n+1}$ satisfies $V(I) = \bigcap_{\alpha \in A} V(F_\alpha)$. This shows that any tropical prevariety can be the underlying set of points of an ideal in $\overline{\mathbb{R}}[\mathbf{x}]$. Moreover, note that any rational half-hyperplane in $\mathbb{R}^{n+1}$ is a tropical prevariety (a fact that we learned from Paolo Tripoli): If $H \subset \mathbb{R}^{n+1}$ is the half-hyperplane given by $\mathbf{a} \cdot \mathbf{x} = c$ and $\mathbf{b} \cdot \mathbf{x} \geq d$, with $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{n+1}$ and $c, d \in \mathbb{R}$, then*

$$H = V(\mathbf{x}^{\mathbf{a}} \oplus c) \cap V(\mathbf{x}^{\mathbf{b}} \oplus (d-c) \odot \mathbf{x}^{\mathbf{a}} \oplus d)$$

(where we might need to clear denominators if some of the entries of $\mathbf{a}$ and $\mathbf{b}$ are negative). If follows that any rational polyhedron in $\mathbb{R}^{n+1}$ of dimension at most n is a tropical prevariety. By intersecting an infinite collection of polyhedra that cut out a non-polyhedral set, we see that the underlying set of points of an arbitrary ideal in $\overline{\mathbb{R}}[\mathbf{x}]$ need not even be polyhedral.

The proof of the following theorem is mainly based on the ideas developed in Section 3, but we omit details due to space constraints.

Theorem 4.2 (Gröbner complex.) *Let $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ be a homogeneous tropical ideal. There is a finite $\mathbb{R}$-rational polyhedral complex $\Sigma(I) \subset \mathbb{R}^{n+1}$, whose support is all $\mathbb{R}^{n+1}$, such that $\mathbf{w}$ and $\mathbf{w}'$ lie in the same cell of $\Sigma(I)$ if and only if $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}'}(I)$. The polyhedral complex $\Sigma(I)$ is called the* Gröbner complex *of I.*

We can now state the main result of this section.

Corollary 4.3 *If $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ is a tropical ideal, its underlying variety $V(I)$ is the support of a finite polyhedral complex.*

Proof: The underlying variety $V(I)$ is the set of vectors $\mathbf{w}$ for which $\mathrm{in}_{\mathbf{w}} I$ contains no monomial, thus $V(I)$ is the union of the cells of the Gröbner complex $\Sigma(I)$ that correspond to monomial-free initial ideals of I. □

The study of the structure of varieties defined by prime tropical ideals is currently work in progress. In particular, it is unclear if, just as in the realizable case, they are pure and balanced polyhedral complexes.

Definition 4.4 (Tropical bases.) *Let $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ be a tropical ideal. We say that the tropical polynomials $F_1, F_2, \ldots, F_l \in I$ form a* tropical basis *for I if they cut out the underlying variety $V(I)$, i.e.,*

$$V(I) = V(F_1) \cap V(F_2) \cap \cdots \cap V(F_l).$$

The fact that the Gröbner complex is a finite polyhedral complex implies that (finite) tropical bases always exist.

Theorem 4.5 *Any tropical ideal $I \subset \overline{\mathbb{R}}[\mathbf{x}]$ has a finite tropical basis.*

Tropical ideals satisfy the following versions of the Nullstellensatz, which are completely analogous to their classical counterparts.

Theorem 4.6 *(*Tropical Nullstellensatz.*) Let $I \subset \overline{\mathbb{R}}[x_1, \dots, x_n]$ be a tropical ideal.*

(a) The variety $V(I) \cap \mathbb{R}^n$ in the tropical torus is empty if and only if I contains a monomial.

(b) The variety $V(I) \subset \overline{\mathbb{R}}^n$ is empty if and only if I is the unit ideal $\langle 0 \rangle$.

(c) If I is homogeneous, the variety $V(I) \subset \overline{\mathbb{R}}^n$ contains no point other than $\vec{\infty}$ if and only if I contains all monomials of degree at least d, for some $d \geq 0$.

The Nullstellensatz allows us to easily classify all maximal tropical ideals of the semiring $\overline{\mathbb{R}}[x_1, \dots, x_n]$.

Example 4.7 *(*Maximal tropical ideals of $\overline{\mathbb{R}}[\mathbf{x}]$.*) We show that maximal tropical ideals of the semiring of tropical polynomials $\overline{\mathbb{R}}[\mathbf{x}] = \overline{\mathbb{R}}[x_1, \dots, x_n]$ are in one to one correspondence with points in $\overline{\mathbb{R}}^n$. In fact, for any $\mathbf{a} \in \overline{\mathbb{R}}^n$, let $J_{\mathbf{a}}$ be the tropical ideal consisting of all polynomials that tropically vanish on $\mathbf{a}$ (see Example 2.3). If a tropical ideal I satisfies $V(I) \neq \emptyset$ then I must be contained in one of the $J_{\mathbf{a}}$. On the other hand, if $V(I) = \emptyset$ then, by the tropical Nullstellensatz, I must be the unit ideal $\langle 0 \rangle$.*

References

[AB07] Federico Ardila and Sara Billey, *Flag arrangements and triangulations of products of simplices*, Advances in Mathematics **214** (2007), no. 2, 495–524.

[DW92] Andreas Dress and Walter Wenzel, *Valuated matroids*, Adv. Math. **93** (1992), no. 2, 214–250.

[GG] Jeffrey Giansiracusa and Noah Giansiracusa, *Equations of tropical varieties*, Preprint. arXiv:1308.0042.

[GP] Dima Grigoriev and Vladimir V. Podolskii, *Tropical effective primary and dual Nullstellensätze*, Preprint. arXiv:1409.6215.

[Mik05] Grigory Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), no. 2, 313–377.

[MR] Diane Maclagan and Felipe Rincón, *Tropical schemes, tropical cycles, and valuated matroids*, Preprint. arXiv:1401.4654.

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015.

[MT01] Kazuo Murota and Akihisa Tamura, *On circuit valuation of matroids*, Adv. in Appl. Math. **26** (2001), no. 3, 192–225.

[Oxl92] James G. Oxley, *Matroid theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, 1992.

[Rin12] Felipe Rincón, *Isotropical linear spaces and valuated Delta-matroids*, J. Combin. Theory Ser. A **119** (2012), no. 1, 14–32.

[Spe08] David Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558.

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Advances in Geometry **4** (2004), no. 3, 389–411.