Almost simplicial polytopes: the lower and upper bound theorems

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Abstract. This is an extended abstract of the full version. We study n-vertex d-dimensional polytopes with at most one nonsimplex facet with, say, $d+s$ vertices, called *almost simplicial polytopes*. We provide tight lower and upper bounds for the face numbers of these polytopes as functions of d, n and s, thus generalizing the classical Lower Bound Theorem by Barnette and Upper Bound Theorem by McMullen, which treat the case $s = 0$. We characterize the minimizers and provide examples of maximizers, for any d.

Résumé. Ceci est un résumé étendu d'une version plus complète. Nous étudions les polytopes de dimension d à n sommets dont au plus une facette n'est pas un simplexe et contient par exemple $d+s$ sommets. Nous appelons de tels polytopes des polytopes presque simpliciaux. Nous établissons des bornes inférieures et supérieures exactes pour le nombre de faces de ces polytopes en fonction de d, n et s, généralisant ainsi les résultats classiques de Barnette sur la borne inférieure et de McMullen sur la borne supérieure dans le cas où $s = 0$. Nous caractérisons les polytopes possédant un nombre de faces minimales et donnons des exemples de polytopes avec un nombre de faces maximal.

Keywords. polytope, f-vector, LBT, UBT, graph rigidity, moment curve

1 Introduction

In 1970 McMullen [14] proved the Upper Bound Theorem (UBT) for *simplicial polytopes*, polytopes with each facet being a simplex, while between 1971 and 1973 Barnette [2, 3] proved the Lower Bound Theorem (LBT) for the same polytopes. Both results are major achievements in the combinatorial theory of polytopes; see, e.g., the books [10, 19] for further details and discussion.

These results can be phrased as follows: let $C(d, n)$ (resp. $S(d, n)$) denote a cyclic (resp. stacked) d-polytope on n vertices, and for a polytope P let $f_i(P)$ denote the number of its i-dimensional faces. Then the classical LBT and UBT read as follows.

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Theorem 1.1 (Classical LBT and UBT) *For any simplicial d-polytope on n vertices, and any $0 \leq i \leq d-1$,*

$$f_i(S(d,n)) \leq f_i(P) \leq f_i(C(d,n)).$$

The numbers $f_i(S(d,n))$ and $f_i(C(d,n))$ are explicit known functions of (d,n,i), to be discussed later.

We generalize the UBT and LBT to the following context: consider a pair (P,F) where P is a polytope, F is a facet of P, and all facets of P different from F are simplices. We call such a polytope P an *almost simplicial polytope* (ASP) and a pair (P,F) an ASP-pair. We will be interested only in the combinatorics of P, thus the ASP-pair (P,F) is equivalent to specifying a regular triangulation of F admitting a lifting of its vertices that leaves the vertices of F fixed; we are interested in the simplicial ball $P' := \partial P - \{F\}$.

Let $\mathcal{P}(d,n,s)$ denote the family of d-polytopes P on n-vertices such that (P,F) is an ASP-pair, where F has $d+s$ vertices ($s \geq 0$). Note that $\mathcal{P}(d,n,0)$ consists of the simplicial d-polytopes on n vertices. In this paper, we define certain polytopes $C(d,n,s), S(d,n,s) \in \mathcal{P}(d,n,s)$, explicitly compute their face numbers, and show the following.

Theorem 1.2 (LBT and UBT for ASP) *For any d,n,s, any polytope $P \in \mathcal{P}(d,n,s)$, and any $0 \leq i \leq d-1$,*

$$f_i(S(d,n,s)) \leq f_i(P) \leq f_i(C(d,n,s)).$$

Further, the polytopes $P \in \mathcal{P}(d,n,s)$ with $f_i(P) = f_i(S(d,n,s))$ for some $0 \leq i \leq d-1$ *are characterized combinatorially, and satisfy the above equality for* all $0 \leq i \leq d-1$.

The characterization of the equality case above generalizes Kalai's result [11] that equality in the classical LBT holds for some $1 \leq i \leq d-1$ iff P is stacked. The polytopes $C(d,n,s)$ form an ASP analog of cyclic polytopes and satisfy a combinatorial Gale-evenness type description of their facets.

Billera and Lee [5] considered the notion of polytope pairs. In particular, their results give tight upper and lower bound theorems for the face numbers of simplicial $(d-1)$-dimensional balls of the "*polytope-antistar*" form; that is, balls of the form $\partial Q - v$, where Q is a simplicial d-polytope and v is a vertex of Q that is deleted. These bounds are given as functions of $d, f_0(\partial Q - v), f_0(Q/v)$, where Q/v denotes the vertex figure of v in Q. For an ASP-pair (P,F), let Q be obtained from P by stacking a pyramid over F with a new vertex v. Then $F \cong Q/v$ and $P' = \partial P - \{F\} = \partial Q - v$. Thus, our balls P' form a subfamily of the balls $\partial Q - v$ considered in [5]. The bounds we obtain in Theorem 1.2 are strictly stronger than those of [5] which apply to all polytope-antistar balls.

Let $f(P) = (1, f_0(P), f_1(P), \cdots, f_{d-1}(P))$ denote the *f-vector* of P, a vector recording the face numbers of P. The following problem naturally arises.

Problem 1.3 *Characterize the pairs of f-vectors $(f(P), f(F))$ for ASP-pairs (P,F).*

A solution would generalize the well known g-theorem characterizing the face numbers of simplicial polytopes, conjectured by McMullen [15] and proved by Billera-Lee [4] (sufficiency) and Stanley [17] (necessity). We leave this general problem to a future study.

The proof of the LBT for ASP and the characterization of the equality cases are based on framework-rigidity arguments (cf. Kalai [11]) and on an adaptation of the well known McMullen-Perles-Walkup reduction (MPW) [11, Sec. 5] to ASP; see Section 3.

The numerical bounds obtained in the UBT for ASP are a special case of a recent result of Adiprasito and Sanyal [1, Thm. 3.10], who proved the bounds for homology balls whose boundary is an induced subcomplex. While their proof relies on machinery from commutative algebra, we provide an elementary proof based on a suitable shelling of P. Further, our construction of maximizers $C(d,n,s)$ is a generalization of cyclic polytopes, based on a suitable variation of the moment curve, and is of independent interest; see Section 4.

2 Preliminaries

For undefined terminology and notation, see [19] for polytopes and complexes, or [11, Sec. 2] for framework rigidity.

2.1 Polytopes and simplicial complexes

The k-dimensional faces of a polyhedral complex Δ are called k-faces, where the empty face has dimension -1. For a simplicial complex Δ of dimension $d-1$, the number $f_k(\Delta)$ is then related to the h-numbers $h_k(\Delta) := \sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{k-i}f_{i-1}(\Delta)$ by

$$f_{k-1}(\Delta) = \sum_{i=0}^{k}\binom{d-i}{k-i}h_i(\Delta). \tag{1}$$

The h-vector of Δ, $(\ldots, h_k, h_{k+1}, \ldots)$, can be considered as an infinite sequence if we let $h_k(\Delta) = 0$ for $k > d$ and $k < 0$. The g-numbers are defined by $g_k(\Delta) = h_k(\Delta) - h_{k-1}(\Delta)$.

For an ASP pair (P, F), where P is d-dimensional, the following version of the Dehn-Somerville equations applies to the complex $P' = \partial P - \{F\}$.

Proposition 2.1 ([9, Thm. 18.3.6], Dehn-Somerville Equations for P') *The h-vector of the simplicial $(d-1)$-ball P' with boundary ∂F satisfies for $k = 0, \ldots, d$*

$$h_k(P') = h_{d-k}(P') + g_k(\partial F). \tag{2}$$

Note that $h_k(P') = 0$ and $h_k(\partial F) = 0$ for $k \geq d$ and $h_{d-1}(\partial F) = 1$.

Note that P' is shellable, by a Bruggesser-Mani line shelling.

The *link* of a face F in Δ is $\mathrm{link}_\Delta(F) := \{T \in \Delta : T \cap F = \emptyset,\ F \cup T \in \Delta\}$, and its *star*, $\mathrm{star}_\Delta(F)$ is the complex $\cup_{F \subseteq T} 2^T$. Thus, using the *join* operator on simplicial complexes, we obtain $2^F * \mathrm{link}_\Delta(F) = \mathrm{star}_\Delta(F)$. The definition of the star extends to polyhedral complexes. For a vertex v in a polytope Q, its *vertex figure* Q/v is a codimension 1 polytope obtained by intersecting Q with a hyperplane H a bit *below* v, so that v is on one side of H and the other vertices of Q are on the other side. If $\mathrm{star}_Q(v)$ is simplicial then the boundary complex of Q/v coincides with $\mathrm{link}_Q(v)$.

A subcomplex K of Δ is *induced* if it contains all the faces in Δ which only involve vertices in K. Note that, for an ASP-pair (P, F), ∂F is an induced subcomplex of P', by convexity.

A polytope is *k-neighborly* if each subset of at most k vertices forms the vertex set of a face. A $\lfloor d/2 \rfloor$-neighborly d-polytope is simply called *neighborly*. A polytope is *k-simplicial* if each k-face is a simplex.

The underlying set $|\mathcal{C}|$ of a polyhedral complex $\mathcal{C}$ is the point set $\cup_{Q\in\mathcal{C}} Q$ of its geometric realization. A *refinement* (or subdivision) of $\mathcal{C}$ is another polyhedral complex $\mathcal{D}$ such that $|\mathcal{D}| = |\mathcal{C}|$ and for any face $F \in \mathcal{D}$ there exists a face $T \in \mathcal{C}$ such that $|F| \subseteq |T|$.

Let G be a proper face of a polytope Q. A point w is *beyond* G (with respect to Q) if (i) w is not on any hyperplane supporting a facet of Q, (ii) w and the interior of Q lie on different sides of any hyperplane supporting a facet containing G, but (iii) on the same side of every other facet-defining hyperplane which does not contain G. For an ASP-pair (P, F) we will consider the simplicial polytope Q obtained as the convex hull of P and a vertex y beyond F.

A simplicial complex Δ is a *homology sphere* (over a fixed field $\mathbf{k}$) if for any face $F \in \Delta$, the homology groups $H_i(\mathrm{link}_\Delta(F); \mathbf{k}) \cong H_i(S^{\dim \Delta - \dim F - 1}; \mathbf{k})$ for all i, where S^j is the j-dimensional sphere. Say Δ is a *homology ball* if $H_i(\mathrm{link}_\Delta(F); \mathbf{k})$ vanishes for $i < \dim \Delta - \dim F - 1$ and is isomorphic to either 0 or $\mathbf{k}$ for $i = \dim \Delta - \dim F - 1$. Furthermore, the boundary complex $\partial\Delta$ of Δ, consisting of all faces F for which $H_{\dim \Delta - \dim F - 1}(\mathrm{link}_\Delta(F); \mathbf{k}) = 0$, is a homology sphere (of codimension 1). In particular, simplicial spheres (resp. balls) are homology spheres (resp. balls).

A polytope is *stacked* if it can be obtained from a simplex by repeatedly taking the convex hull with a vertex beyond some facet. A homology sphere is *stacked* if it is combinatorially isomorphic to the boundary complex of a stacked polytope.

2.2 Rigidity

We mostly follow the presentation in Kalai's [11]. Let $G = (V, E)$ be a graph, and $d(a, b)$ denote Euclidean distance between points a and b in Euclidean space. A d-embedding $f : V \to \mathbf{R}^d$ is called *rigid* if there exists an $\epsilon > 0$ such that if $g : V \to \mathbf{R}^d$ satisfies $d(f(v), g(v)) < \epsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$. G is called *generically d-rigid* if the set of its rigid d-embeddings is open and dense in the topological vector space of all of its d-embeddings. Given a d-embedding $f : V \to \mathbf{R}^d$, a *stress* of f is a function $w : E \to \mathbf{R}$ such that for every vertex $v \in V$

$$\sum_{u:\{v,u\} \in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

The stresses of f form a vector space, called the *stress space*. Its dimension is the same for generic d-embeddings (namely, for an open and dense set in the space of all d-embeddings of G). A graph G is called *generically d-stress free* if this dimension is zero.

If a generic $f : V \to \mathbf{R}^d$ is rigid, then $f_1(G) \geq df_0(G) - \binom{d+1}{2}$. Thus, if Δ is a simplicial complex of dimension $d-1$ whose 1-skeleton is generically d-rigid, then $f_1(\Delta) \geq df_0(\Delta) - \binom{d+1}{2}$, and $g_2(\Delta)$ is the dimension of the stress space of any generic embedding. Based on these observations for Δ the boundary of a simplicial d-polytope with $d \geq 3$, and more general complexes, Kalai [11] extended the LBT and characterized the minimizers.

For a d-polytope P with a simplicial 2-skeleton, the toric $g_2(P)$ equals $g_2(\partial P) := f_1(P) - df_0(P) + \binom{d+1}{2}$, and by a result of Alexandrov (cf. Whiteley [18]), it equals the dimension of the stress space of the 1-skeleton of P. For our LBT for ASP, we will need the following very special case of Kalai's monotonicity[(i)], which Kalai proved using rigidity arguments.

Theorem 2.2 (Kalai's Monotonicity [12, Thm. 4.1], weak form) *Let $d \geq 4$, P a d-polytope with a simplicial 2-skeleton, and F a facet of P. Then*

$$g_2(P) \geq g_2(F).$$

Equivalently, $f_1(P) - f_1(F) \geq (df_0(P) - \binom{d+1}{2}) - ((d-1)f_0(F) - \binom{d}{2})$.

[(i)] Kalai's monotonicity conjecture on the toric g-polynomials, asserting that $g(P) \geq g(F)g(P/F)$ coefficientwise for any face F of P, was first proved for rational polytopes by Braden and MacPherson [7]. By the combinatorial intersection homology, it is now known to hold in full generality; cf. [6].

3 A lower bound theorem for almost simplicial polytopes

Recall that a simplicial d-polytope is called *stacked* if it can be obtained from a d-simplex by repeated *stacking*, namely, adding a vertex beyond a facet and taking the convex hull. While stacked d-polytopes on n vertices, denoted $S(d,n)$, may have different combinatorial structures, they all have the same f-vector, given by

$$f_k(S(d,n)) = \phi_k(d,n) := \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } k = 1,\ldots,d-2 \\ (d-1)n-(d+1)(d-2) & \text{for } k = d-1. \end{cases}$$

For any integers $d \geq 3$, $s \geq 0$ and $n \geq d+s+1$, let F be a stacked $(d-1)$-polytope with $d+s$ vertices. Construct a pyramid over F and then stack $n-d-s-1$ times over facets of the resulting polytope which are different from F to obtain a polytope $S(d,n,s)$ in $\mathcal{P}(d,n,s)$. One easily computes the f-vector of $S(d,n,s)$, since refining F by its (unique) stacked triangulation refines the boundary complex of $S(d,n,s)$ to a stacked simplicial sphere with f-vector $f(S(d,n))$. We obtain

$$f(S(d,n,s)) = f(S(d,n)) - (0,0,\cdots,0,s,s).$$

Note that for $n \geq s+4$, any $P \in \mathcal{P}(3,n,s)$ has f-vector $f(P) = (1,n,3n-6-s,2n-4-s) = f(S(3,n,s))$. We are ready to state the LBT for ASP; its minimizers will be characterized later.

Theorem 3.1 (LBT for ASP) *Let $d \geq 3$, $s \geq 0$, $n \geq d+s+1$. Then for any $P \in \mathcal{P}(d,n,s)$ and $1 \leq i \leq d-1$ we have*

$$f_i(S(d,n,s)) \leq f_i(P).$$

Proof: We proceed by induction on d, the case $d=3$ was verified above. Let $d \geq 4$. By a result of Whiteley [18], the 1-skeleton of P is generically d-rigid, hence $f_1(P) \geq \phi_1(d,n)$, and by the MPW reduction, $f_i(P) \geq \phi_i(d,n)$ for all $2 \leq i \leq d-3$ as well; see [11, Thm. 12.2] [(ii)].

Denote by (P,F) the ASP-pair, and by $\deg_P(v)$ the degree of a vertex v in the 1-skeleton of P. We now prove the inequality for the facets, by a variation of the MPW reduction. Note that the vertex figure P/v in P of any vertex $v \in \operatorname{vert} F$ is an ASP (with $\deg_P(v)$ vertices), while for any vertex $v \in \operatorname{vert} P \backslash \operatorname{vert} F$ P/v is a simplicial polytope; cf. [8, Thm. 11.5]. Furthermore, for a vertex $v \in \operatorname{vert} F$, letting $s_v := \deg_F(v) - (d-1) \geq 0$ gives $P/v \in \mathcal{P}(d-1, \deg_P(v), s_v)$.

Double counting the number of pairs (v,A) for a vertex v in a facet A of P, we obtain the following inequalities:

$$\begin{aligned}
d(f_{d-1}(P)-1)+(d+s) &= \sum_{v\in \operatorname{vert} P} f_{d-2}(\operatorname{link}_P(v)) \\
&\geq \sum_{v\in \operatorname{vert} P\backslash \operatorname{vert} F} ((d-2)\deg_P(v) - d(d-3)) + \sum_{v \in \operatorname{vert} F} ((d-2)\deg_P(v) - d(d-3) - s_v) \\
&= 2(d-2)f_1(P) - d(d-3)f_0(P) - 2f_1(F) + (d-1)(d+s) \\
&\geq 2(d-2)\left[df_0(P) - \binom{d+1}{2}\right] - d(d-3)f_0(P) - 2\left[(d-1)f_0(F) - \binom{d}{2}\right] + (d-1)(d+s) \\
&= d(d-1)f_0(P) - d(d+1)(d-2) - s(d-1),
\end{aligned}$$

(ii) Kalai's theorem contains a typo. It includes the case $i=k$, while it holds only for $i<k$, where P is k-simplicial. Our ASP P is $(d-2)$-simplicial.

where the first inequality is by the induction hypothesis and the second inequality is by Kalai's monotonicity Theorem 2.2 and the LBT inequality for $f_1(P)$. Comparing the LHS with the RHS gives

$$f_{d-1}(P) \geq \phi_{d-1}(d,n) - s.$$

The inequality for $f_{d-2}(P)$ follows from the inequality for $f_{d-1}(P)$ by double counting. Since any ridge in P is contained in exactly two facets, counting the number of pairs (R, A) for a ridge R in a facet A of P, we obtain that

$$2f_{d-2}(P) = d(f_{d-1}(P) - 1) + f_{d-2}(F).$$

Applying the classical LBT to the simplicial polytope F with $f_0(F) = d + s$, we get

$$2f_{d-2}(P) \geq d(f_{d-1}(P) - 1) + (d-2)(d+s) - d(d-3),$$

and applying the lower bound for $f_{d-1}(P)$ yields, after dividing both sides by 2, the desired lower bound $f_{d-2}(P) \geq \phi_{d-2}(d,n) - s$. □

We now turn our attention to characterizing the minimizers of Theorem 3.1. We start with some terminology and background.

A proper subset A of the vertices of a d-polytope P is called a *missing k-face* of P if the cardinality of A is $k+1$, the simplex on A is not a face of P, but for any proper subset B of A the simplex on B is a face of P. If A is a missing $(d-1)$-face of P then adding the simplex A cuts P into two d-polytopes P_1, P_2, glued along the simplex A. We denote this operation by $P = P_1 \# P_2$. Repeating this procedure on each P_i until no piece P_i contains a missing $(d-1)$-face results in a decomposition $P = P_1 \# P_2 \# \cdots \# P_t$, where intersections along missing $(d-1)$-faces of P define a tree whose vertices are the P_i's. Note that for $d \geq 3$ a decomposition of P as above is uniquely defined; just insert all the missing $(d-1)$-faces. Call such a decomposition the *prime decomposition* of P, and call each P_i a *prime factor* of P. Denote by Δ_P the polyhedral complex defined by the prime decomposition of P. Then a simplicial d-polytope P is stacked iff all its prime factors are d-simplices. This definition immediately extends to polyhedral spheres where the operation # corresponds to the topological connected sum.

We start with the characterization of the minimizers for the easier case $d > 4$.

Theorem 3.2 (Characterization of minimizers for $d > 4$) *Let $d > 4$ and $P \in \mathcal{P}(d,n,s)$. Let Δ_F be the polyhedral complex corresponding to the prime decomposition of F, and let Δ be the refinement of the boundary complex ∂P of P obtained by refining F by Δ_F. Assume there is some $1 \leq i \leq d-1$ for which $f_i(P) = f_i(S(d,n,s))$. Then, all prime factors in the prime decomposition of Δ are d-simplices. In particular, $f(P) = f(S(d,n,s))$.*

Proof: By the MPW reduction and the variation of it we used in the proof of Theorem 3.1, it is enough to consider the case $i = 1$. From Kalai's monotonicity (Theorem 2.2) and our assumption $g_2(P) = 0$, it follows that $g_2(F) = 0$. As F is simplicial of dimension ≥ 4, Kalai's [11, Thm. 1.1(ii)] says that F is stacked, thus Δ is a simplicial $(d-1)$-sphere. Since $g_2(\Delta) = 0$, by [11, Thm. 1.1(ii)] again, Δ is stacked, as desired. In particular, $f(P) = f(S(d,n,s))$. □

For $d = 4$, F need not be stacked. For example, the pyramid over any simplicial 3-polytope is a minimizer. We obtain the following characterization of minimizers.

Theorem 3.3 (Characterization of minimizers for $d = 4$**)** *Let $P \in \mathcal{P}(4, n, s)$, and keep the notation of Theorem 3.2. Assume there is some $1 \le i \le d-1$ for which $f_i(P) = f_i(S(d, n, s))$. Then, the prime factors in the prime decomposition of Δ are either d-simplices with no facet contained in $|F|$, or pyramids over prime factors of F.*

In order to prove this theorem we first need to show generic d-rigidity for the 1-skeleton of a much larger class of complexes. Let $\mathcal{C}_k$ be the family of homology k-balls Δ such that:

- the induced subcomplex $\Delta[I]$ on the set of internal vertices I has a connected 1-skeleton, and
- for any edge e in the boundary complex $\partial\Delta$, there exists a 2-simplex T, $e \subset T$, such that T has a vertex in I.

Note that any homology k-ball Δ whose boundary $\partial\Delta$ is an induced subcomplex is in $\mathcal{C}_k$. In particular, for $P \in \mathcal{P}(d, n, s)$, the simplicial complex $P' = \partial P - \{F\}$ is in $\mathcal{C}_{d-1}$.

Lemma 3.4 *Let $d \ge 4$. The 1-skeleton of any $\Delta \in \mathcal{C}_{d-1}$ is generically d-rigid. Thus, $f_1(\Delta) \ge df_0(\Delta) - \binom{d+1}{2}$.*

The proof is similar to Kalai's proof of the classical LBT [11] and is omitted.

Proof of Theorem 3.3: Consider a prime factor L of Δ. Then L is a 4-polytope with a generically 4-rigid 1-skeleton. As $g_2(P) = 0$, the 1-skeleton of L, denoted by G, must be generically 4-stress free. Thus, $g_2(L) = 0$.

If L does not contain a facet in F, then L is simplicial, with $g_2(L) = 0$, hence is stacked by [11, Thm. 1.1]. Being also prime, L is a 4-simplex.

Assume then that L contains a facet F'' contained in $|F|$, so (L, F'') is an ASP-pair. If L has a unique vertex outside F'', then L is a pyramid over a prime factor of F and we are done. Assume the contrary, so there is an edge $vu \in G$ with $v, u \notin F''$ (for concreteness, taking v, u to be the highest two vertices of L above the hyperplane of F works).

First we show that vu satisfies the link condition $\mathrm{link}_L(v) \cap \mathrm{link}_L(u) = \mathrm{link}_L(vu)$, which guarantees that contracting the edge vu in the simplicial complex $\partial L - \{F''\}$ results in $\tilde{\Delta} \in \mathcal{C}_3$; see e.g.[16, Prop.2.4][(iii)]. Indeed, if vu fails the link condition it means that vu is contained in a missing face M, with 3 or 4 vertices. Now, M cannot have 4 vertices as L is prime. If $M = vuz$ then uz is an edge of L *not* in $\mathrm{link}_L(v)$. Since $\mathrm{link}_L(v)$ is a homology 2-sphere (thus, a simplicial 2-sphere), its 1-skeleton is generically 3-rigid. Consequently, the 1-skeleton of $\mathrm{star}_L(v)$ is generically 4-rigid, and adding vu to it yields a 4-stress in G, a contradiction.

Let m be the number of vertices in the cycle $\mathrm{link}_L(vu)$, then $f_1(\tilde{\Delta}) = f_1(L) - m - 1$ and $f_0(\tilde{\Delta}) = f_0(L) - 1$, which implies that $g_2(L) = g_2(\tilde{\Delta}) + (m - 3)$.

If $m > 3$, then applying Lemma 3.4 to $\tilde{\Delta}$ yields $g_2(L) > 0$, a contradiction. So assume $m = 3$.

Denote by x, y, z the vertices of $\mathrm{link}_L(vu)$. If the triangle $xyz \in L$, then, as L is prime, both tetrahedra $xyzv, xyzu$ are faces of L, so L is the 4-simplex $xyzuv$, a contradiction (as it has a facet F'' in F).

We are left to consider the case $xyz \notin L$. The argument here is inspired by Barnette [2, Thm. 2]. In this case, the 3-ball formed by the join $vu * \partial(xyz)$ is an induced subcomplex of $\partial L - \{F''\}$. Now replace it by $\partial(vu) * xyz$ (this is a bistellar move) to obtain from $\partial L - \{F''\}$ the complex Δ''. Clearly Δ'' is a

(iii) To apply [16, Prop. 2.4], phrased for homology spheres, simply cone the boundary of the homology ball Δ to obtain a homology sphere.

homology 3-ball, and any edge on its boundary is part of a 2-simplex with an internal vertex (just take the same one as in $\partial L - \{F''\}$). To show $\Delta'' \in \mathcal{C}_3$ we are left to show that the graph on the internal vertices I of Δ'' is connected. Assume not, namely removing the edge uv disconnects the induced graph on I in $\partial L - \{F''\}$. In particular, $x, y, z \in F''$. But $xyz \notin L$, so xyz is a missing face of F'', contradicting that F'' is a prime factor of F.

We conclude that $\Delta'' \in C_3$, thus, by Lemma 3.4, $\Delta'' \cup \{vu\}$ has a nonzero 4-stress, so $g_2(L) > 0$, a contradiction. The proof is then complete. □

Remark 3.5 *The above shows that, for any $k \geq 3$, the lower bounds of Theorem 3.1 are valid for any complex in $\mathcal{C}_k$, and the minimizers in $\mathcal{C}_k$ are exactly the complexes $\partial P - \{F\}$ described in Theorems 3.2 and 3.3.*

4 An upper bound theorem for almost simplicial polytopes

Throughout this section, we let $P \in \mathcal{P}(d, n, s)$ denote an almost simplicial polytope, (P, F) the ASP-pair, and $P' = \partial P \setminus \{F\}$ the corresponding (shellable) simplicial $(d-1)$-ball.

4.1 ASP generalization of cyclic polytopes

The *moment curve* in $\mathbf{R}^d$ is defined by $t \mapsto (t, t^2, \ldots, t^d)$ for $t \in \mathbf{R}^d$. We consider related curves $x(t) = (t, t^2, \ldots, t^{d-r}, p_1(t), \ldots, p_r(t))$, where $p_i(t)$ are continuous functions in t for $i = 1, \ldots, r$. Later, a special choice of the curve $x(t)$ and points on it will give our maximizer polytope $C(d, n, s)$.

We let $V(t_1, \ldots, t_d)$ denote the *Vandermonde determinant* on variables $t_1, \ldots, t_d$.

$$V(t_1, \ldots, t_d) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_d \\ t_1^2 & t_2^2 & \cdots & t_d^2 \\ \vdots & \vdots & \ldots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_d^{d-1} \end{vmatrix} = \prod_{1 \leq i < j \leq d} (t_j - t_i).$$

Lemma 4.1 *Consider the curve $x(t)$. Then the following holds:*

1. *Any $d - r + 1$ points on the curve $x(t)$ are affinely independent.*
2. *For any n distinct numbers $t_1, \ldots, t_n$, the polytope $Q = \operatorname{conv}(\{x(t_1), \ldots, x(t_n)\})$ is $(d - r - 1)$-simplicial.*
3. *The polytope Q is $\lfloor (d - r)/2 \rfloor$-neighbourly.*

The proof is similar to [10, Sec. 4.7], and is omitted.

Consider the curve $y(t) = (t, t^2, \ldots, t^{d-1}, p(t))$, where

$$p(t) := (n-1)^{(t-1)(d-1)} t(t+1) \cdots (t+d+s-1),$$

and n and s are fixed integers with $n > d + s$ and $s \geq 0$. Let $C(d, n, s) := \operatorname{conv}(\{y(t_1), \ldots, y(t_n)\})$, where $t_i = -s - d + i$ for $i = 1, \ldots, n$. Also, let $T = \{t_i : i = 1, \ldots, n\}$, $I = \{t_i : i = 1, \ldots, d + s\}$ and $y(S) := \{y(t_i) : t_i \in S\}$ for $S \subset T$.

The following proposition collects a number of properties of the d-polytope $C(d, n, s)$.

Proposition 4.2 *The d-polytope $C(d,n,s)$ ($n > d+s$) satisfies the following properties.*

1. $C(d,n,s) \in \mathcal{P}(d,n,s)$.
2. **Gale's evenness condition***: A d-subset S_d of* $\operatorname{vert} C(d,n,s)$ *such that $S_d \not\subset I$ forms a simplex facet iff, for any two elements $u, v \in T \setminus S_d$, the number of elements of S_d between u and v on the curve $y(t)$ is even.*

Proof: (1) We first show that the first $d+s$ vertices span a facet F. Let $y = (y_1, \ldots, y_d) \in \mathbf{R}^d$ and let

$$D((t_1, t_2, \ldots, t_d); y) := \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ t_1 & t_2 & \cdots & t_d & y_1 \\ t_1^2 & t_2^2 & \cdots & t_d^2 & y_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_d^{d-1} & y_{d-1} \\ p(t_1) & p(t_2) & \cdots & p(t_d) & y_d \end{vmatrix}.$$

Let $D(y) := D((t_1, t_2, \ldots, t_d); y)$ and consider the hyperplane $H_D := \{y \in \mathbf{R}^d : D(y) = 0\}$. The points $y(t_i)$ ($i = 1, \ldots, d+s$) are all contained in H_D, by vanishing of the last row of $D(y(t_i))$. Let $y(t^*) \in \operatorname{vert} C(d,n,s) \setminus y(I)$, then $D(y(t^*)) = p(t^*)V(t_1, \ldots, t_d) > 0$ since $p(t^*) > 0$ and $V(t_1, \ldots, t_d) > 0$. Thus, F is a facet of $C(d,n,s)$.

We now show that every other facet is a simplex. Consider any $(d+1)$-set $\{t_{i_1} < \ldots < t_{i_d} < t_{i_{d+1}} = t^*\} \subset T$ not contained in I. Thus, $t^* \in T \setminus I$. Consider the determinant $E(y) := D((t_{i_1}, t_{i_2}, \ldots, t_{i_d}); y)$.

The hyperplane $H_E := \{y \in \mathbf{R}^d : E(y) = 0\}$ contains all the points $y(t_{i_j})$ ($j = 1, \ldots, d$). We need to show that $E(y(t^*)) \neq 0$.

Note that $p(t) = 0$ for $t \in I$ and $p(t) > 0$ for $t \in T \setminus I$. Also, note that $|t_a - t_b| \leq n-1$ for $t_a, t_b \in [-s-d+1, -s-d+n]$. For the sake of clarity assume d is odd; the case of even d is analogous. Computing $E(y(t^*))$ by expanding w.r.t. the last row gives

$$\begin{aligned} &\big(p(t^*)V(t_{i_1}, \ldots, t_{i_d}) - p(t_{i_d})V(t_{i_1}, \ldots, t_{i_{d-1}}, t^*)\big) + \cdots \\ &\quad + \left(p(t_{i_2})V(t_{i_1}, t_{i_3} \ldots, t^*) - p(t_{i_1})V(t_{i_2}, \ldots, t^*)\right). \end{aligned}$$

The definition of $p(t)$ implies that each pair-summand is nonnegative and the first pair-summand is positive, and so the determinant is positive. Indeed, for $j > 1$, if $p(t_{i_j}) = 0$ then also $p(t_{i_{j-1}}) = 0$ and the corresponding pair-summand vanishes. Otherwise, let $V(j) := V(t_{i_1}, \ldots t_{i_{j-1}}, t_{i_{j+1}}, \ldots, t_{i_{d+1}})$ for short. Then,

$$p(t_{i_j})V(j) \geq (n-1)^{(d-1)(t_{i_j}-1)} \prod_{l=0}^{d+s-1} (t_{i_{j-1}} + l) \frac{V(j-1)}{(n-1)^{d-1}} \geq p(t_{i_{j-1}})V(j-1).$$

This completes the proof of the first assertion.

(2) Consider a set $S_d = \{t_{i_1} < \ldots < t_{i_d}\} \not\subset I$. Let $t^* \in T$, $t_{i_{j-1}} < t^* < t_{i_j}$ (include also the cases $t^* < t_{i_1}$ with $j = 1$ and $t_{i_d} < t^*$ where we put $j = d+1$). From the above reasoning we see that if the column $y(t^*)$ in the determinant $E(y(t^*))$ is placed between the columns $y(t_{i_{j-1}})$ and

$y(t_{i_j})$ then the resulting determinant is positive. To achieve this, we swap $d-j+1$ times the column $y(t^*)$, which gives that the sign of $E(y(t^*))$ is $(-1)^{d-j+1}$. Consequently, on the curve $y(t)$, between $[-s-d+1,-s-d+n]$, the determinant $E(y(t^*))$ changes sign whenever the variable passes through one of the values t_{i_j} $(i=1,\dots,d)$, and we are done. □

A polytope $C(d,n,s)$ will be called *almost cyclic*. Having established in 4.1 that $C(d,n,s)$ is $\lfloor(d-1)/2\rfloor$-neighbourly, we can compute its h-vector, in steps. Recall that $P' = \partial P \setminus \{F\}$.

Proposition 4.3 *Let $P \in \mathcal{P}(d,n,s)$ be $\lfloor(d-1)/2\rfloor$-neighbourly, and (P,F) the ASP-pair. Then,*

$$h_k(P') = \binom{n-d-1+k}{k}, \qquad \text{if } 0 \le k \le \lfloor(d-1)/2\rfloor;$$
$$h_{d-k}(P') = \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, \qquad \text{if } 1 \le k \le \lfloor(d-1)/2\rfloor.$$

The proof basically uses the fact that $f_{k-1}(P') = \binom{n}{k}$ for $k \le \lfloor(d-1)/2\rfloor$, and the Dehn-Sommerville relations (2); we omit the details.

Observe that, for even d, being $\lfloor(d-1)/2\rfloor$-neighbourly does not determine the value of $h_{d/2}(P')$. With the help of Gale's evenness condition we can compute the number of facets of $C(d,n,s)$, and together with 4.3 and (1), we can compute $h_{d/2}(C(d,n,s))$ for any even d as well. Let $C' := C(d,n,s) - \{F\}$.

Proposition 4.4 *For the ASP-pair $(C(d,n,s),F)$ with d even, consider the simplicial ball C'. Then*

$$f_{d-1}(C') = \left(\binom{n-d/2-1}{d/2} + \sum_{i=0}^{d/2-1} 2\binom{n-d-1+i}{i}\right) - \binom{s+d/2}{d/2}.$$

Proof: The counting argument for the facets of C', based on Gale evenness, goes as in the proof of the number of facets of cyclic polytopes (cf. [19, Cor. 8.28]), with the difference that we discard the Gale d-tuples formed solely by the first $d+s$ vertices, thus we discard exactly $\binom{s+d/2}{d/2}$ of them. □

Corollary 4.5 *The h-numbers of C' are given by:*

$$h_k(C') = \binom{n-d-1+k}{k}, \qquad \text{if } 0 \le k \le \lfloor(d-1)/2\rfloor;$$
$$h_{d-k}(C') = \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, \qquad \text{if } 1 \le k \le \lfloor d/2\rfloor.$$

Proof: The case of odd d was already established by 4.3 since $C(d,n,s)$ is $\lfloor(d-1)/2\rfloor$-neighbourly. For the case of even d it remains to compute $h_{d/2}(P)$. Equating the corresponding expression in Proposition 4.4 with the expression of f_{d-1} in (1), after substituting the known values of h_k for $k \ne d/2$, gives

$$\begin{aligned} h_{d/2}(C') &= \binom{n-d/2-1}{d/2} + \sum_{i=0}^{d/2-1} \binom{s+i-1}{i} - \binom{s+d/2}{d/2} \\ &= \binom{n-d/2-1}{d/2} - \binom{s+d/2-1}{d/2}, \end{aligned}$$

as desired. $\square$

4.2 An upper bound theorem for almost simplicial polytopes

We are now in a position to state an upper bound theorem for almost simplicial polytopes $P \in \mathcal{P}(d, n, s)$.

Theorem 4.6 (UBT for ASP) *Any almost simplicial polytope $P \in \mathcal{P}(d, n, s)$ satisfies*

$$h_k(P') \leq \binom{n-d-1+k}{k}, \qquad \textit{if } 0 \leq k \leq \lfloor (d-1)/2 \rfloor; \tag{3}$$

$$h_{d-k}(P') \leq \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, \qquad \textit{if } 1 \leq k \leq \lfloor d/2 \rfloor. \tag{4}$$

Thus,

$$f_{i-1}(P) \leq f_{i-1}(C(d, n, s) \quad \textit{for } i = 1, 2, \ldots, d,$$

for the almost cyclic d-polytope $C(d, n, s)$. Equality for some f_{i-1} with $\lfloor (d-1)/2 \rfloor \leq i \leq d$ implies that P is $\lfloor (d-1)/2 \rfloor$-neighbourly.

Proof of 4.6 via [1, Thm. 3.10]: The inequalities on $h_k(P')$ hold for $0 \leq k \leq d-1$ by [1, Thm. 3.10], as P' is a special case of a homology ball whose boundary is an induced subcomplex. From Corollary 4.5 and (1) the inequality $f_{i-1}(P) \leq f_{i-1}(C(d, n, s)$ follows. Equality for some f_{i-1} with $d \geq i \geq \lfloor (d-1)/2 \rfloor$ implies, by (1), the equality $h_k(P') = \binom{n-d-1+k}{k}$ for all $0 \leq k \leq \lfloor (d-1)/2 \rfloor$, and thus, again by (1), that P is $\lfloor (d-1)/2 \rfloor$-neighbourly. $\square$

Remark 4.7 (More maximizers.) *As is the case with neighborly polytopes, we expect that there are many combinatorially distinct ASPs achieving the upper bounds in the UBT for ASP. We obtained another such construction, based on a certain perturbation of the Cayley polytopes constructed by Karavelas and Tzanaki [13, Sec.5]; the details are omitted from this extended abstract.*

Remark 4.8 *We produced an alternative and elementary proof of the UBT for ASP, via shelling. Our proof follows ideas from the proof of the classical UBT by McMullen, cf. [19, Sec.8.4], and from a recent work of Karavelas and Tzanaki [13], and includes a special property of shellings of ASPs. For space limits we omit the details and refer to the full arXiv version.*

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